

Rouquier Complexes are Functorial over Braid Cobordisms

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Abstract

Using the diagrammatic calculus for Soergel bimodules developed by B. Elias and M. Khovanov, we show that Rouquier complexes are functorial over braid cobordisms. We explicitly describe the chain maps which correspond to movie move generators.

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1 Introduction

For some time, the category of Soergel bimodules, here called \mathcal{SC} , has played a significant role in the study of representation theory, while more recently strong connections between \mathcal{SC} and knot theory have come to light. Originally introduced by Soergel in [11], \mathcal{SC} is an equivalent but more combinatorial description of a certain category of Harish-Chandra modules over a semisimple lie algebra \mathfrak{g} . The added simplicity of this formulation comes from the fact that \mathcal{SC} is just a full monoidal subcategory of graded R -bimodules, where R is a polynomial ring equipped with an action of the Weyl group of \mathfrak{g} . Among other things, Soergel gave an isomorphism between the Grothendieck ring of \mathcal{SC} and the Hecke algebra \mathcal{H} associated to \mathfrak{g} , where the Kazhdan-Lusztig generators b_i of \mathcal{H} lift to bimodules B_i which are easily described. The full subcategory generated monoidally by these bimodules B_i is here called \mathcal{SC}_1 , and the category including all grading shifts and direct sums of objects in \mathcal{SC}_1 is called \mathcal{SC}_2 . It then turns out that \mathcal{SC} is actually the idempotent closure of \mathcal{SC}_2 , which reduces the study of \mathcal{SC} to the study of these elementary bimodules B_i and their tensors. For more on Soergel bimodules and their applications to representation theory see [12, 13, 14].

An important application of Soergel bimodules was discovered by Rouquier in [10], where he observes that one can construct complexes in \mathcal{SC}_2 which satisfy the braid relations modulo homotopy. To the i^{th} overcrossing (resp. undercrossing) in the braid group Rouquier associates a complex, which has R in homological degree 0 and B_i in homological degree -1 (resp. 1). Giving the homotopy equivalence classes of invertible complexes in \mathcal{SC}_2 the obvious group structure under tensor product, this assignment extends to a homomorphism from the braid group. Using this, one can define an action of the braid group on the homotopy category of \mathcal{SC}_2 , where the endofunctor associated to a crossing is precisely taking the tensor product with its associated complex.

Following his work with L. Rozansky on matrix factorizations and link homology in [5], Khovanov produced an equivalent categorification [4] of the HOMFLY-PT polynomial utilizing Rouquier's work. To a braid one associates its Rouquier complex, which naturally has two gradings: the homological grading, and the internal grading of Soergel bimodules. Then, taking the Hochschild homology of each term in the complex, one gets a complex which is triply graded (the third grading is the Hochschild homological grading). Khovanov showed that, up to degree shifts, this construction yields an equivalent triply-graded complex to the one produced by the reduced version of the Khovanov-Rozansky HOMFLY-PT link homology for the closure of the braid (see [4] and [5] for more details).

Many computations of HOMFLY-PT link homology were done by B. Webster [16], and by J. Rasmussen in [8] and [9]. In the latter paper [9], Rasmussen showed that given a braid presentation of a link, for every $n \in \mathbb{N}$ there exists a spectral sequence with E^1 -term its HOMFLY-PT homology and the E^∞ -term its $sl(n)$ homology. This was a spectacular development in understanding the structural properties of these theories, and has also proven very useful in computation (see for example [7]).

One key aspect of the original Khovanov-Rozansky theory is that it gives rise to a projective functor. The braid group can actually be realized as the isomorphism classes of objects in the category of braid cobordisms. This category, while having a topological definition, is equivalent to a combinatorially defined category, whose objects are braid diagrams, and whose morphisms are called *movies* (see Carter-Saito, [1]). For instance, performing a Reidemeister 3 move on a braid diagram would give an equivalent element of the braid group,

but gives a distinct object in the braid cobordism category; however, the R3 move itself is a movie which gives the isomorphism between those two objects. It was shown in [5] that for each movie between braids one can associate a chain map between their triply-graded complexes. This assignment was known to be *projectively functorial*, meaning that the relations satisfied amongst movies in the braid cobordism category are also satisfied by their associated chain maps, up to multiplication by a scalar. Scalars take their value in \mathbb{Q} , the ring over which Khovanov-Rozansky theory is defined. However, these chain maps are not explicitly described even in the setting of Khovanov-Rozansky theory, and the maps they correspond to in the Soergel bimodule context are even more obscure. A more general discussion of braid group actions, including this categorification via Rouquier complexes, and their extensions to projective actions on the category of braid cobordisms can be found in [6].

Recently, in [3], the first author in conjunction with Mikhail Khovanov gave a presentation of the category \mathcal{SC}_1 in terms of generators and relations. Moreover, it was shown that the entire category can be drawn graphically, thanks to the biadjointness and cyclicity properties that the category possesses. Each B_i is assigned a color, and a tensor product is assigned a sequence of colors. Morphisms between tensor products can be drawn as certain colored graphs in the plane, whose boundaries on bottom and top are the sequence of colors associated to the source and target. Composition and tensor product of morphisms correspond to vertical and horizontal concatenation, respectively. Morphisms are invariant under isotopy of the graph embedding, and satisfy a number of other relations, as described herein. In addition to providing a presentation, this graphical description is useful because one can use pictures to encapsulate a large amount of information; complicated calculations involving compositions of morphisms can be visualized intuitively and written down suffering only minor headaches.

Because of the simplicity of the diagrammatic calculus, we were able to calculate explicitly the chain maps which correspond to each generating cobordism in the braid cobordism category, and check that these chain maps satisfy the same relations that braid cobordisms do. The general proofs are straightforward and computationally explicit, performable by any reader with patience, time, and colored chalk. While we use some slightly more sophisticated machinery to avoid certain incredibly lengthy computations, the machinery is completely unnecessary. This makes the results of Rouquier and Khovanov that much more concrete, and implies the following new result.

Theorem 1. *There is a functor F from the category of combinatorial braid cobordisms to the category of complexes in \mathcal{SC}_2 up to homotopy, lifting Rouquier's construction (i.e. such that F sends crossings to Rouquier complexes).*

Soergel bimodules are generally defined over certain fields \mathbb{k} in the literature, because one is usually interested in Soergel bimodules as a categorification of the Hecke algebra, and in relating indecomposable bimodules to the Kazhdan-Lusztig canonical basis. However, we invite the reader to notice that the diagrammatic construction in [3] can be made over any ring, and in particular over \mathbb{Z} . In fact, all our proofs of functoriality still work over \mathbb{Z} . We discuss this in detail in section 5.2. In the subsequent paper, we plan to use the work done here to define HOMFLY-PT and $sl(n)$ -link homology theories over \mathbb{Z} , a construction which is long overdue. We also plan to investigate the Rasmussen spectral sequence in this context.

At the given moment there does not exist a diagrammatic calculus for the higher Hochschild homology of Soergel bimodules. Some insights have already been obtained, although a full

understanding had yet to emerge. We plan to develop the complete picture, which should hopefully give an explicit and easily computable description of functoriality in the link homology theories discussed above.

The organization of this paper is as follows. In Section 2 we go over all the previous constructions that are relevant to this paper. This includes the Hecke algebra, Soergel's categorification \mathcal{SC} , the graphical presentation of \mathcal{SC} , the combinatorial braid cobordism category, and Rouquier's complexes which link \mathcal{SC} to braids. In Section 2.6 we describe the conventions we will use in the remainder of the paper to draw Rouquier complexes for movies. In Section 3 we define the functor from the combinatorial braid cobordism category to the homotopy category of \mathcal{SC} , and in Section 4 we check the movie move relations to verify that our functor is well-defined. These checks are presented in numerical order, not in logical order, but a discussion of the logical dependency of the proofs, and of the simplifications that are used, can be found in Section 4.1. Section 5 contains some useful statements for the interested reader, but is not strictly necessary. Some additional light is shed on the generators and relations of \mathcal{SC} in Section 5.1, where it is demonstrated how the relations arise naturally from movie moves. In Section 5.2 we briefly describe how one might construct the theory over \mathbb{Z} , so that future papers may use this result to define link homology theories over arbitrary rings.

2 Constructions

2.1 The Hecke Algebra

The Hecke algebra \mathcal{H} of type A_∞ has a presentation as an algebra over $\mathbb{Z}[t, t^{-1}]$ with generators b_i , $i \in \mathbb{Z}$ and the *Hecke relations*

$$b_i^2 = (t + t^{-1})b_i \tag{1}$$

$$b_i b_j = b_j b_i \text{ for } |i - j| \geq 2 \tag{2}$$

$$b_i b_{i+1} b_i + b_{i+1} = b_{i+1} b_i b_{i+1} + b_i. \tag{3}$$

For any subset $I \subset \mathbb{Z}$, we can consider the subalgebra $\mathcal{H}(I) \subset \mathcal{H}$ generated by b_i , $i \in I$, which happens to have the same presentation as above. Usually only finite I are considered.

We write the monomial $b_{i_1} b_{i_2} \cdots b_{i_d}$ as $b_{\underline{i}}$ where $\underline{i} = i_1 \dots i_d$ is a finite sequence of indices; by abuse of notation, we sometimes refer to this monomial simply as \underline{i} . If \underline{i} is as above, we say the monomial has *length* d . We call a monomial *non-repeating* if $i_k \neq i_l$ for $k \neq l$. The empty set is a sequence of length 0, and $b_\emptyset = 1$.

Let ω be the t -antilinear anti-involution which fixes b_i , i.e. $\omega(t^a b_{\underline{i}}) = t^{-a} b_{\sigma(\underline{i})}$ where σ reverses the order of a sequence. Let $\epsilon: \mathcal{H} \rightarrow \mathbb{Z}[t, t^{-1}]$ be the $\mathbb{Z}[t, t^{-1}]$ -linear map which is uniquely specified by $\epsilon(xy) = \epsilon(yx)$ for all $x, y \in \mathcal{H}$ and $\epsilon(b_{\underline{i}}) = t^d$, whenever \underline{i} is a non-repeating sequence of length d . Let $(,): \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{Z}[t, t^{-1}]$ be the map which sends $(x, y) \mapsto \epsilon(\omega(x)y)$. Via the inclusion maps, these structures all descend to each $\mathcal{H}(I)$ as well.

We say $i, j \in \mathbb{Z}$ are *adjacent* if $|i - j| = 1$, and are *distant* if $|i - j| \geq 2$.

For more details on the Hecke algebra in this context, see [3].

2.2 The Soergel Categorification

In [11], Soergel introduced a monoidal category categorifying the Hecke algebra for a finite Weyl group W of type A . We will denote this category by $\mathcal{SC}(I)$, or by \mathcal{SC} when I is irrelevant. Letting V be the geometric representation of W over a field \mathbb{k} of characteristic $\neq 2$, and R its coordinate ring, the category \mathcal{SC} is given as a full additive monoidal subcategory of graded R -bimodules (whose objects are now commonly referred to as Soergel bimodules). This category is not abelian, for it lacks images, kernels, and the like, but it is idempotent closed. In fact, \mathcal{SC} is given as the idempotent closure of another full additive monoidal subcategory \mathcal{SC}_1 , whose objects are called Bott-Samuelson modules. The category \mathcal{SC}_1 is generated monoidally over R by objects B_i , $i \in I$, which satisfy

$$B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\} \quad (4)$$

$$B_i \otimes B_j \cong B_j \otimes B_i \text{ for distant } i, j \quad (5)$$

$$B_i \otimes B_j \otimes B_i \oplus B_j \cong B_j \otimes B_i \otimes B_j \oplus B_i \text{ for adjacent } i, j. \quad (6)$$

The Grothendieck group of $\mathcal{SC}(I)$ is isomorphic to $\mathcal{H}(I)$, with the class of B_i being sent to b_i , and the class of $R\{1\}$ being sent to t .

One useful feature of this categorification is that it is easy to calculate the dimension of Hom spaces in each degree. Let $\text{HOM}(M, N) \stackrel{\text{def}}{=} \bigoplus_{m \in \mathbb{Z}} \text{Hom}(M, N\{m\})$ be the graded vector space (actually an R -bimodule) generated by homogeneous morphisms of all degrees. Let $B_{\underline{i}} \stackrel{\text{def}}{=} B_{i_1} \otimes \cdots \otimes B_{i_d}$. Then $\text{HOM}(B_{\underline{i}}, B_{\underline{j}})$ is a free left R -module, and its graded rank over R is given by $(b_{\underline{i}}, b_{\underline{j}})$.

For two subsets $I \subset I' \subset \mathbb{Z}$, the categories $\mathcal{SC}(I)$ and $\mathcal{SC}(I')$ are embedded in bimodule categories over different rings $R(I)$ and $R(I')$, but there is nonetheless a faithful inclusion of categories $\mathcal{SC}(I) \rightarrow \mathcal{SC}(I')$. This functor is not full: the size of R itself will grow, and $\text{HOM}(B_\emptyset, B_\emptyset) = R$. However, the graded rank over R does not change, since the value of ϵ and hence $(,)$ does not change over various inclusions. Effectively, the only difference in Hom spaces under this inclusion functor is base change on the left, from $R(I)$ to $R(I')$.

As a result of this, most calculations involving morphisms between Soergel bimodules will not depend on which I we work over. When I is infinite, the ring R is no longer Noetherian, and we do not wish to deal with such cases. However, the categories $\mathcal{SC}(I)$ over arbitrary finite I will all work essentially the same way. A slightly more rigorous graphical statement of this property is forthcoming. In particular, the calculations we do for the Braid group on m strands will also work for the braid group on $m + 1$ strands, and so forth.

2.3 Soergel Diagrammatics

In [3], the category \mathcal{SC}_1 was given a diagrammatic presentation by generators and relations, allowing morphisms to be viewed as isotopy classes of certain graphs. We review this presentation here, referring the reader to [3] for more details. We will first deal with the case where $W = S_{n+1}$, or where $I = \{1, 2, \dots, n\}$, and then discuss what the inclusions of categories from the previous section imply for the general setting.

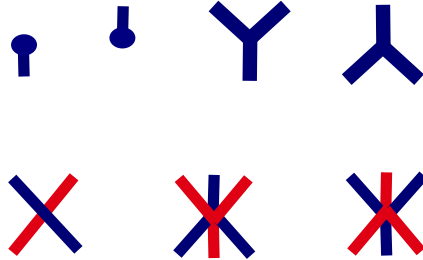
Remark. Technically, [3] gave the presentation for a slightly different category, which we temporarily call \mathcal{SC}'_1 . The category presented here is a quotient of \mathcal{SC}'_1 by the central

morphism corresponding to e_1 , the first symmetric polynomial. This is discussed briefly in Section 4.5 of [3]. Moreover, \mathcal{SC}'_1 is also a faithful extension of \mathcal{SC}_1 , so that the main results of this paper apply to the extension as well. We use \mathcal{SC}_1 instead because it is the “minimal” category required for our results (no extensions are necessary), and because it streamlines the presentation. We leave it as an exercise to see that the definition of \mathcal{SC}_1 below agrees with the e_1 quotient of the category defined in [3].

The first subtlety to be addressed is that \mathcal{SC}_1 is only equivalent to the e_1 quotient of \mathcal{SC}'_1 when one is working over a base ring \mathbb{k} where $n + 1$ is invertible. Otherwise, the quotient of \mathcal{SC}'_1 is still a non-trivial faithful extension.

For a discussion of the advantages to using \mathcal{SC}'_1 , see Section 5.2.

An object in \mathcal{SC}_1 is given by a sequence of indices \underline{i} , which is visualized as d points on the real line \mathbb{R} , labelled or “colored” by the indices in order from left to right. Sometimes these objects are also called $B_{\underline{i}}$. Morphisms are given by pictures embedded in the strip $\mathbb{R} \times [0, 1]$ (modulo certain relations), constructed by gluing the following generators horizontally and vertically:



For instance, if “blue” corresponds to the index i and “red” to j , then the lower right generator is a morphism from jij to iji . The generating pictures above may exist in various colors, although there are some restrictions based on adjacency conditions.

We can view a morphism as an embedding of a planar graph, satisfying the following properties:

1. Edges of the graph are colored by indices from 1 to n .
2. Edges may run into the boundary $\mathbb{R} \times \{0, 1\}$, yielding two sequences of colored points on \mathbb{R} , the top boundary \underline{i} and the bottom boundary \underline{j} . In this case, the graph is viewed as a morphism from \underline{j} to \underline{i} .
3. Only four types of vertices exist in this graph: univalent vertices or “dots”, trivalent vertices with all three adjoining edges of the same color, 4-valent vertices whose adjoining edges alternate in colors between i and j distant, and 6-valent vertices whose adjoining edges alternate between i and j adjacent.

The degree of a graph is $+1$ for each dot and -1 for each trivalent vertex. 4-valent and 6-valent vertices are of degree 0. The term *graph* henceforth refers to such a graph embedding.

By convention, we color the edges with different colors, but do not specify which colors match up with which $i \in I$. This is legitimate, as only the various adjacency relations between colors are relevant for any relations or calculations. We will specify adjacency for

all pictures, although one can generally deduce it from the fact that 6-valent vertices only join adjacent colors, and 4-valent vertices join only distant colors.

As usual in a diagrammatic category, composition of morphisms is given by vertical concatenation, and the monoidal structure is given by horizontal concatenation.

In writing the relations, it will be useful to introduce a pictures for the “cup” and “cap”:

$$\begin{aligned} \cup &= \Upsilon \\ \cap &= \gamma \end{aligned} \tag{7}$$

We then allow \mathbb{k} -linear sums of graphs, and apply the relations below to obtain our category \mathcal{SC}_1 . Some of these relations are redundant. For a more detailed discussion of the remarks in the remainder of this section, see [3].

$$\hook = | = \cap \tag{8}$$

$$\cap = \mid = \cup \tag{9}$$

$$\Upsilon = \gamma = \cap \tag{10}$$

$$\hook \cap = \times = \cap \cup \tag{11}$$

$$\cup \cap = \times = \cap \cup \tag{12}$$

Remark. The relations (8) through (12) together imply that the morphism specified by a particular graph embedding is independent of the isotopy class of the embedding. We could have described the category more simply by defining a morphism to be an isotopy class of a certain kind of planar graph. However, it is useful to understand that these “isotopy relations” exist, because they will appear naturally in the study of movie moves (see Section 5.1).

Other relations are written in a format which already assumes that isotopy invariance is given. Some of these relations contain horizontal lines, which cannot be constructed using the generating pictures given; nonetheless, such a graph is isotopic to a number of different pictures which are indeed constructible, and it is irrelevant which version you choose, so the relation is unambiguous.

$$\Upsilon = \times \tag{13}$$

Remark. Relation (13) effectively states that a certain morphism is invariant under 90 degree rotation. To simplify drawings later on, we often draw this morphism as follows:



Note that morphisms will still be isotopy invariant with this convention.

Here are the remainder of the one color relations.

$$\text{Y-junction} = | \text{line} = \text{Y-junction} \quad (14)$$

$$\text{Loop} = 0 \quad (15)$$

$$\text{Line with dot} + | \text{Line with dot} = 2 | \text{Line with dot} \quad (16)$$

In the following relations, the two colors are distant.

$$\text{Crossing of distant colors} = || \quad (17)$$

$$\text{Crossing of distant colors with dot} = \text{Crossing of distant colors} \quad (18)$$

$$\text{Y-junction of distant colors} = \text{Y-junction of distant colors} \quad (19)$$

$$| \text{Line with dot} = | \text{Line with dot} \quad (20)$$

In this relation, two colors are adjacent, and both distant to the third color.

$$\text{Crossing of adjacent colors} = \text{Crossing of adjacent colors} \quad (21)$$

In this relation, all three colors are mutually distant.

$$\text{Crossing of adjacent colors with dot} = \text{Crossing of adjacent colors with dot} \quad (22)$$

Remark. Relations (17) thru (22) indicate that any part of the graph colored i and any part of the graph colored j “do not interact” for i and j distant. That is, one may visualize sliding the j -colored part past the i -colored part, and it will not change the morphism. We call this the *distant sliding property*.

In the following relations, the two colors are adjacent.

$$\text{Crossing of adjacent colors} = \text{Y-junction of adjacent colors} + \text{Loop of adjacent colors} \quad (23)$$

$$\text{Red Blue Red} = \text{Red over Blue over Red} - \text{Red under Blue under Red} \quad (24)$$

$$\text{Diagram 1} = \text{Diagram 2} \quad (25)$$

$$\text{Red double dot on Blue line} - \text{Blue double dot on Red line} = -\text{Blue boundary dot} + \text{Red boundary dot} = \frac{1}{2}(\text{Blue double dot} - \text{Red double dot}) \quad (26)$$

The last equality in (26) is implied by (16), so it is not necessary to include as a relation. In this final relation, the colors have the same adjacency as $\{1, 2, 3\}$.

$$\text{Diagram 1} = \text{Diagram 2} \quad (27)$$

Remark. Because of isotopy invariance, the object B_i in \mathcal{SC}_1 is self-biadjoint. In particular, instead of viewing the graph in $\mathbb{R} \times [0, 1]$ as a morphism from \underline{i} to \underline{j} , we could twist it around and view it in the lower half plane (with no bottom boundary) as a morphism from \emptyset to $\underline{i}\sigma(\underline{j})$. Thus, we need only investigate morphisms from \emptyset to \underline{i} , to determine all Hom spaces.

Remark. There is a functor from this category into the category of R -bimodules, sending a line colored i to B_i and each generator to an appropriate bimodule map. The functor gives an equivalence of categories between this graphically defined category and the subcategory \mathcal{SC}_1 of R -bimodules mentioned in the previous section, so the use of the same name is legitimate.

We refer to any connected component of a graph which is a dot connected directly to the boundary as a *boundary dot*, and to any component equal to two dots connected by an edge as a *double dot*.

Remark. Relations (16), (20), and (26) are collectively called *dot slides*. They indicate how one might attempt to move a double dot from one region of the graph to another.

The following theorem and corollary are the most important results from [3], and the crucial fact which allows all other proofs to work.

Theorem 2. Consider a morphism $\phi: \underline{i} \rightarrow \emptyset$, and suppose that the index i appears in \underline{i} zero times (respectively, once). Then ϕ can be rewritten as a linear combination of graphs, for which each graph has the following property: the only edges of the graph colored i are included in double dots (respectively, as well as a single boundary dot connecting to \underline{i}), and moreover, all these double dots are in the leftmost region of the graph. This result may be obtained simultaneously for multiple indices i . We could also have chosen the rightmost region for the slide.

Corollary 3. *The space $\text{HOM}_{\mathcal{SC}_1}(\emptyset, \emptyset)$ is the free commutative polynomial ring generated by f_i , the double dot colored i , for various $i \in I$. This is a graded ring, with the degree of f_i is 2.*

Remark. Another corollary of the more general results in [3] is that, when a color only appears twice in the boundary one can (under certain conditions on other colors present) reduce the graph to a form where that color only appears in a line connecting the two boundary appearances (and double dots as usual). In particular, if no color appears more than twice on the boundary, then under certain conditions one can reduce all graphs to a form that has no trivalent vertices, and hence all morphisms have nonnegative degree. We will use this fact to help check movie moves 8 and 9, in whose contexts the appropriate conditions do hold.

The proof of this theorem involves using the relations to reduce a single color at a time within a graph (while doing arbitrary things to the other colors). Once a color is reduced to the above form, the remainder of the graph no longer interacts with that color. Then we repeat the argument with another color on the rest of the graph, and so on and so forth.

Remark. There is a natural identification of the polynomial ring of double dots and the coordinate ring R of the geometric representation. Because of this, a combination of double dots is occasionally referred to as a polynomial. Placing double dots in the lefthand or righthand region of a diagram will correspond to the left and right action of R on Hom spaces.

Remark. Now we are in a position to see how the inclusion $\mathcal{SC}_1(I) \subset \mathcal{SC}_1(I')$ behaves. Let \underline{i} and \underline{j} be objects in $\mathcal{SC}_1(I)$, and k an index in $I' \setminus I$. Applying Theorem 2 to the color k , we can assume that in $\mathcal{SC}_1(I')$ all morphisms from \underline{i} to \underline{j} will be (linear combinations of) graphs where k only appears in double dots on the left. Doing this to each color in $I' \setminus I$, we will have a collection of double dots next to a morphism which only uses colors in I . Therefore the map $\text{HOM}_{\mathcal{SC}(I)}(\underline{i}, \underline{j}) \otimes \mathbb{k}[f_k, k \in I' \setminus I] \rightarrow \text{HOM}_{\mathcal{SC}(I')}(\underline{i}, \underline{j})$ is surjective. In fact, it is an isomorphism. We say that the inclusion functor is *fully faithful up to base change*. Of course, this result does not make it any easier to take a graph, which may have an arbitrarily complicated k -colored part, and reduce it to the simple form where k only appears in double dots on the left.

If we wished to define $\mathcal{SC}_1(I)$ for some $I \subset \{1, \dots, n\}$, the correct definition would be to consider graphs which are only colored by indices in I . With this definition, inclusion functors are still fully faithful up to base change.

Now we see where the isomorphisms (4) through (6) come from. To begin, we have the following implication of (16):

$$\parallel = \frac{1}{2} (\text{X} + \text{X}) \quad (28)$$

We let \mathcal{SC}_2 be the category formally containing all direct sums and grading shifts of objects in \mathcal{SC}_1 , but whose morphisms are forced to be degree 0. Then (28) expresses the direct sum decomposition

$$B_i \otimes B_i = B_i\{1\} \oplus B_i\{-1\}$$

since it decomposes the identity id_{ii} as a sum of two orthogonal idempotents, each of which is the composition of a projection and an inclusion map of the appropriate degree. If one does not wish to use non-integral coefficients, and an adjacent color is present, then the following implication of (26) can be used instead; this is again a decomposition of id_{ii} into orthogonal idempotents.

$$\parallel = (\text{X}_{\text{blue}} + \text{X}_{\text{red}}) - \text{X}_{\text{red}} \quad (29)$$

Relation (17) expresses the isomorphism

$$B_i \otimes B_j = B_j \otimes B_i$$

for i and j distant.

The category \mathcal{SC} is the Karoubi envelope, or idempotent completion, of the category \mathcal{SC}_2 . Recall that the Karoubi envelope of a category \mathcal{C} has as objects pairs (B, e) where B is an object in \mathcal{C} and e an idempotent endomorphism of B . This object acts as though it were the “image” of this projection e , and in an additive category behaves like a direct summand. For more information on Karoubi envelopes, see Wikipedia.

The two color variants of relation (24) together express the direct sum decompositions

$$B_i \otimes B_{i+1} \otimes B_i = C_i \oplus B_i \quad (30)$$

$$B_{i+1} \otimes B_i \otimes B_{i+1} = C_i \oplus B_{i+1}. \quad (31)$$

Again, the identity $\text{id}_{i(i+1)i}$ is decomposed into orthogonal idempotents, where the first idempotent corresponds to a new object C_i in the idempotent completion, appearing as a summand in both $i(i+1)i$ and $(i+1)i(i+1)$. Technically, we get two new objects, corresponding to the idempotent in $B_{i(i+1)i}$ and the idempotent in $B_{(i+1)i(i+1)}$, but these two objects are isomorphic, so by abuse of notation we call them both C_i .

We will primarily work within the category \mathcal{SC}_2 . However, since this includes fully faithfully into \mathcal{SC} , all calculations work there as well.

2.4 Braids and Movies

In this paper we always use the combinatorial braid cobordism category as a replacement for the topological braid cobordism category, since they are equivalent but the former is more convenient for our purposes. See Carter and Saito [1] for more details.

The category of $(n+1)$ -stranded braid cobordisms can be defined as follows. The objects are arbitrary sequences of braid group generators O_i , $1 \leq i \leq n$, and their inverses $U_i = O_i^{-1}$. These sequences can be drawn using braid diagrams on the plane, where O_i is an overcrossing (the $i+1^{\text{st}}$ strand crosses over the i^{th} strand) and U_i is an undercrossing. Objects have a monoidal structure given by concatenation of sequences. A *movie* is a finite sequence of transformations of two types:

I. Reidemeister type moves, such as

$$\tau_1 O_i U_i \tau_2 \leftrightarrow \tau_1 \tau_2,$$

$$\tau_1 O_i O_j \tau_2 \leftrightarrow \tau_1 O_j O_i \tau_2 \text{ for distant } i, j$$

$$\tau_1 O_i O_{i+1} O_i \tau_2 \leftrightarrow \tau_1 O_{i+1} O_i O_{i+1} \tau_2.$$

where τ_1 and τ_2 are arbitrary braid words.

II. Addition or removal of a single O_i or U_i from a braid word

$$\tau_1 \tau_2 \leftrightarrow \tau_1 O_i^{\pm 1} \tau_2.$$

These transformations are known as *movie generators*. Morphisms in this category will consist of movies modulo *locality moves*, which ensure that the category is a monoidal category, and certain relations known as *movie moves* (it is common also to refer to locality moves as movie moves). The movie moves can be found in figures 1 and 2. Movie moves 1 – 10 are composed of type I transformations and 11 – 14 each contains a unique type II move. We denote the location of the addition or removal of a crossing in these last 4 movies by little black triangles. There are many variants of each of these movies: one can change the relative height of strands, can reflect the movie horizontally or vertically, or can run the movie in reverse. We refer the reader to Carter and Saito [1], section 3.

Recall that the combinatorial cobordism category is monoidal. Locality moves merely state that if two transformations are performed on a diagram in locations that do not interact (they do not share any of the same crossings) then one may change the order in which the transformations are performed. Any potential functor from the combinatorial cobordism category to a monoidal category \mathcal{C} which preserves the monoidal structure will automatically satisfy the locality moves. Because of this, we need not mention the locality moves again.

Definition 4. *Given a braid diagram P (or an object in the cobordism category), the diagram \overline{P} is given by reversing the sequence defining P , and replacing all overcrossings with undercrossings and vice versa.*

Note that \overline{P} is the inverse of P in the group generated freely by crossings, and hence in the braid group as well.

Again, we refer the reader to [1] for more details on the combinatorial braid cobordism category.

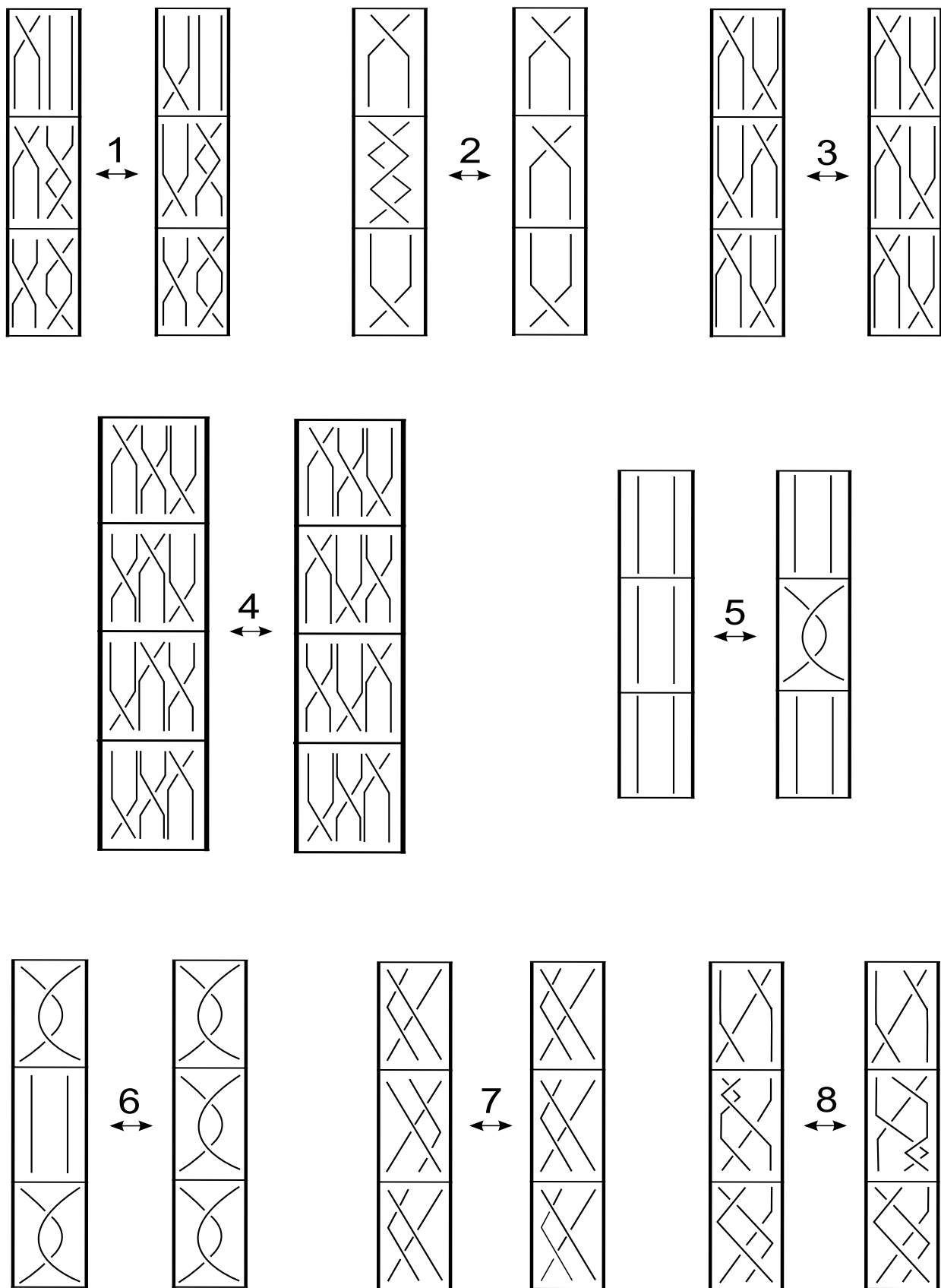


Figure 1: Braid movie moves 1 – 8

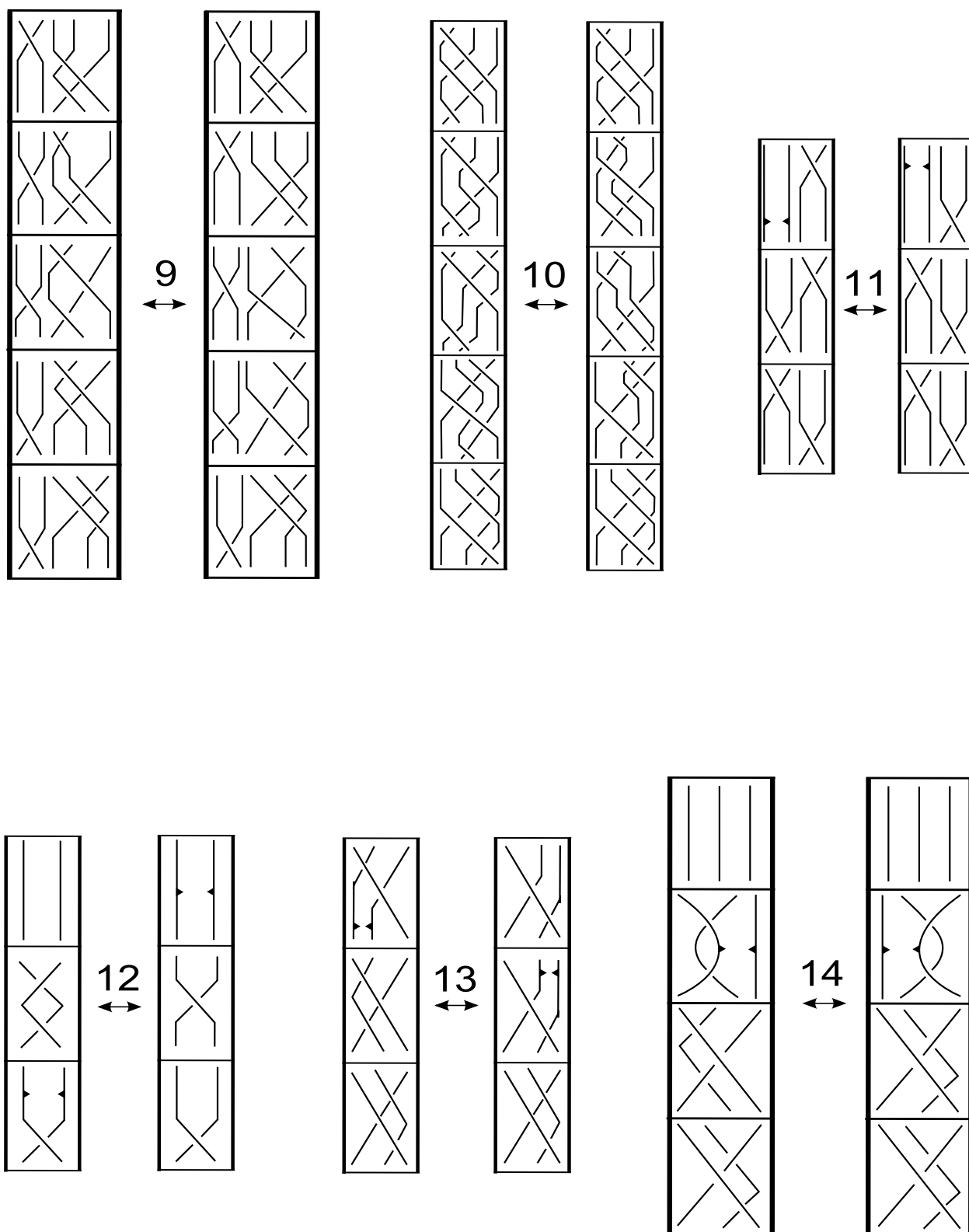


Figure 2: Braid movie moves 9 – 14

2.5 Rouquier Complexes

Rouquier defined a braid group action on the homotopy category of complexes in \mathcal{SC}_2 (see [10]). To the i^{th} overcrossing, he associated a complex $B_i\{1\} \longrightarrow B_\emptyset$, and to the undercrossing, $B_\emptyset \longrightarrow B_i\{-1\}$. In each case, B_\emptyset is in homological degree 0. Drawn graphically, these complexes look like:

$$\begin{array}{c} \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right] : \quad 0 \longrightarrow \underset{-1}{\begin{array}{c} | \{1\} \\ \hline \end{array}} \xrightarrow{\quad \textcolor{blue}{\bullet} \quad} \underset{\underline{0}}{\textcolor{blue}{\bullet}} \longrightarrow 0 \\[2em] \left[\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right] : \quad 0 \longrightarrow \underset{\underline{0}}{\textcolor{blue}{\bullet}} \xrightarrow{\quad \textcolor{blue}{\bullet} \quad} \underset{\underline{1}}{\begin{array}{c} | \{-1\} \\ \hline \end{array}} \longrightarrow 0 \end{array}$$

Figure 3: Rouquier complex for right and left crossings

We are using a (blue) dot here as a place holder for empty space.

To a braid one associates the tensor product of the complexes for each crossing. He showed in [10] that the braid relations hold amongst these complexes.

In [4], Khovanov showed that taking Hochschild cohomology of these complexes yields an invariant of the link which closes off the braid in question, and that this link homology theory is in fact identical to one already constructed by Khovanov and Rozansky in [5]. It was shown in [6] that Rouquier’s association of complexes to a braid is actually *projectively functorial*. In other words, to each movie between braids, there is a map of complexes, and these maps satisfy the movie move relations (modulo homotopy) up to a potential sign. This was not done by explicitly constructing chain maps, but instead used the formal consequences of the previously-defined link homology theory. It was known that in many cases the composed map would be an isomorphism, and that this categorification could be done over \mathbb{Z} (see [6]), where the only isomorphisms are ± 1 , hence the proof of projective functoriality.

The discussion of the previous sections shows that it is irrelevant which braid group we work in, because adding extra strands just corresponds to an inclusion functor which is “fully faithful after base change”. In particular, when computing the space of chain maps modulo homotopy between two complexes, we need not worry about the number of strands available, except to keep track of our base ring. Hence calculations are effectively local.

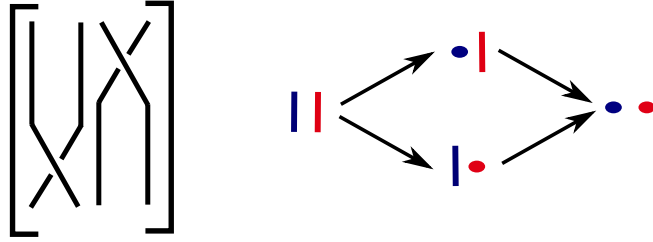
2.6 Conventions

These are the conventions we use to draw Rouquier complexes henceforth.

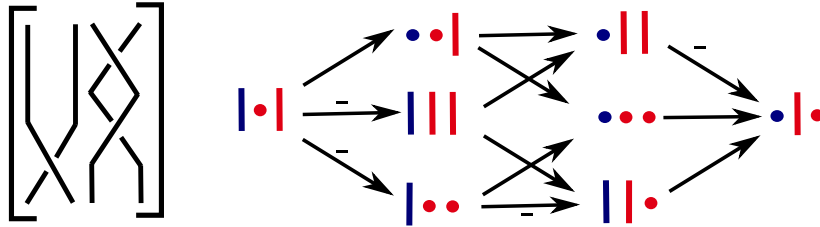
We use a colored circle to indicate the empty graph, but maintain the color for reasons of sanity. It is immediately clear that in the complex associated to a tensor product of d Rouquier complexes, each summand will be a sequence of k lines where $0 \leq k \leq d$ (interspersed with colored circles, but these represent the empty graph so could be ignored).

Each differential from one summand to another will be a “dot” map, with an appropriate sign.

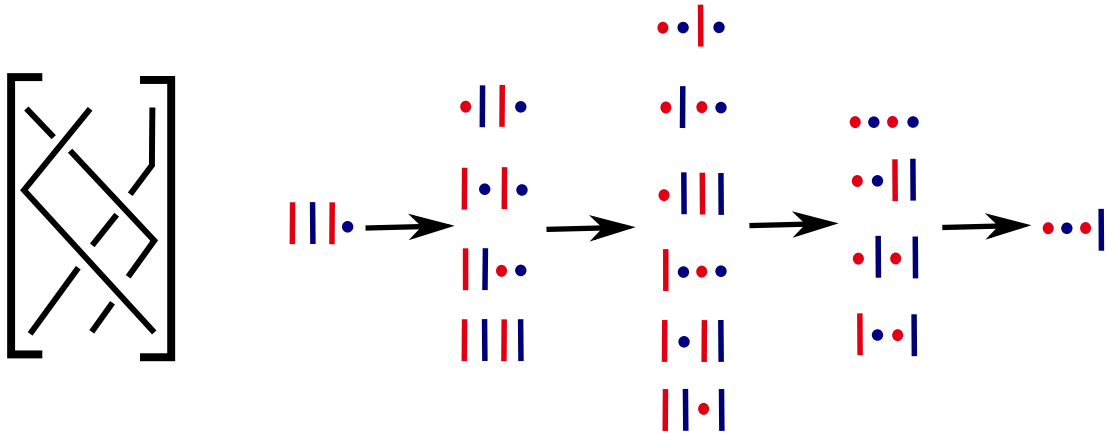
1. The dot would be a map of degree 1 if B_i had not been shifted accordingly. In \mathcal{SC}_2 , all maps must be homogeneous, so we could have deduced the degree shift in B_i from the degree of the differential. Because of this, it is not useful to keep track of various degree shifts of objects in a complex. We will draw all the objects without degree shifts, and all differentials will therefore be maps of graded degree 1 (as well as homological degree 1). It follows from this that homotopies will have degree -1, in order to be degree 0 when the shifts are put back in. One could put in the degree shifts later, noting that B_\emptyset always occurs as a summand in a tensor product exactly once, with degree shift 0.
2. Similarly, one need not keep track of the homological dimension. B_\emptyset will always occur in homological dimension 0.
3. We will use blue for the index associated to the leftmost crossing in the braid, then red and dotted orange for other crossings, from left to right. The adjacency of these various colors is determined from the braid.
4. We read tensor products in a braid diagram from bottom to top. That is, in the following diagram, we take the complex for the blue crossing, and tensor by the complex for the red crossing. Then we translate this into pictures by saying that tensors go from left to right. In other words, in the complex associated to this braid, blue always appears to the left of red.



5. One can deduce the sign of a differential between two summands using the Liebnitz rule, $d(ab) = d(a)b + (-1)^{|a|}ad(b)$. In particular, since a line always occurs in the basic complex in homological dimension ± 1 , the sign on a particular differential is exactly given by the parity of lines appearing to the left of the map. For example,



6. When putting an order on the summands in the tensored complex, we use the following standardized order. Draw the picture for the object of smallest homological degree, which we draw with lines and circles. In the next homological degree, the first summand has the first color switched (from line to circle, or circle to line), the second has the second color switched, and so forth. In the next homological degree, two colors will be switched, and we use the lexicographic order: 1st and 2nd, then 1st and 3rd, then 1st and 4th... then 2nd and 3rd, etc. This pattern continues.



3 Definition of the Functor

We extend Rouquier's complexes to a functor F from the combinatorial braid cobordism category to the category of chain complexes in \mathcal{SC}_2 modulo homotopy. Rouquier already defined how the functor acts on objects, so it only remains to define chain maps for each of the movie generators, and check the movie move relations.

There are four basic types of movie generators: birth/death of a crossing, slide, Reidemeister 2 and Reidemeister 3.

- **Birth and Death generators**

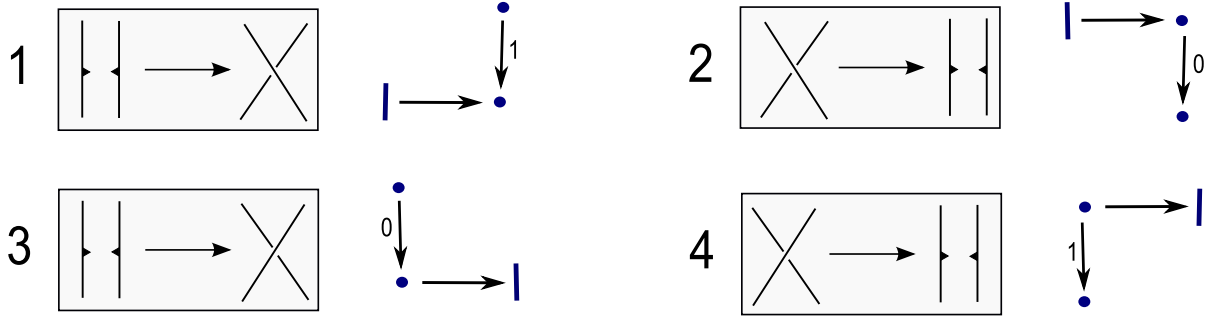


Figure 4: Birth and Death of a crossing generators

- **Reidemeister 2 generators**

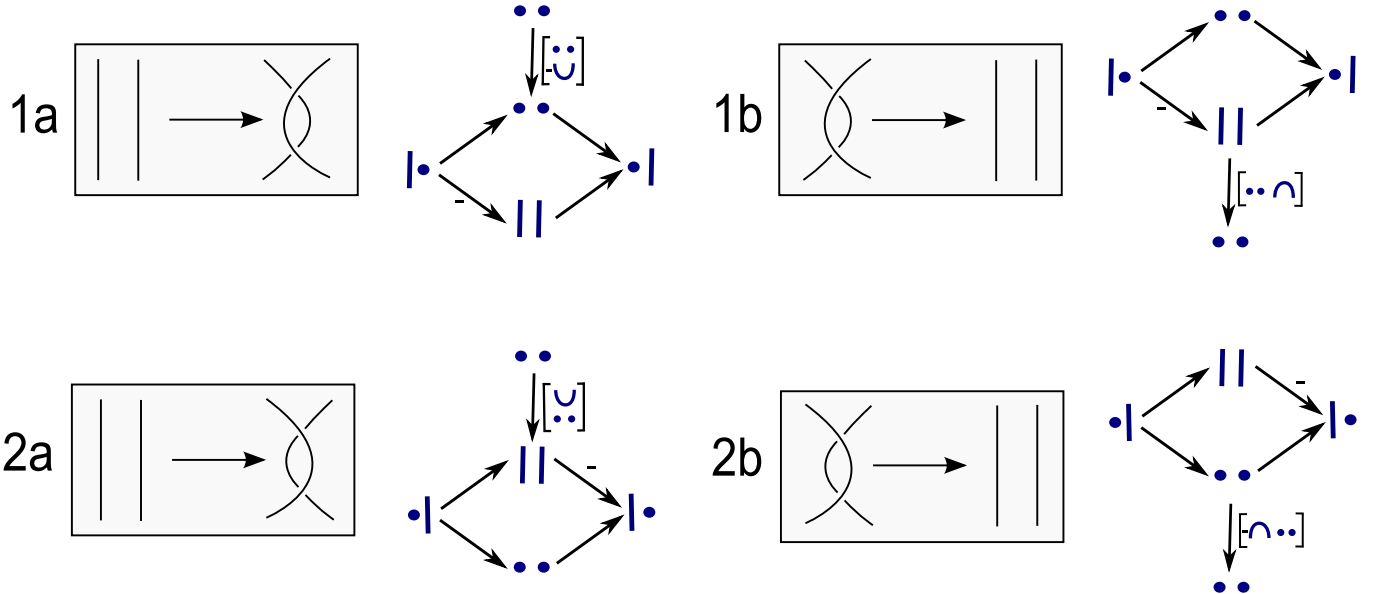


Figure 5: Reidemeister 2 type movie move generators

- Slide generators

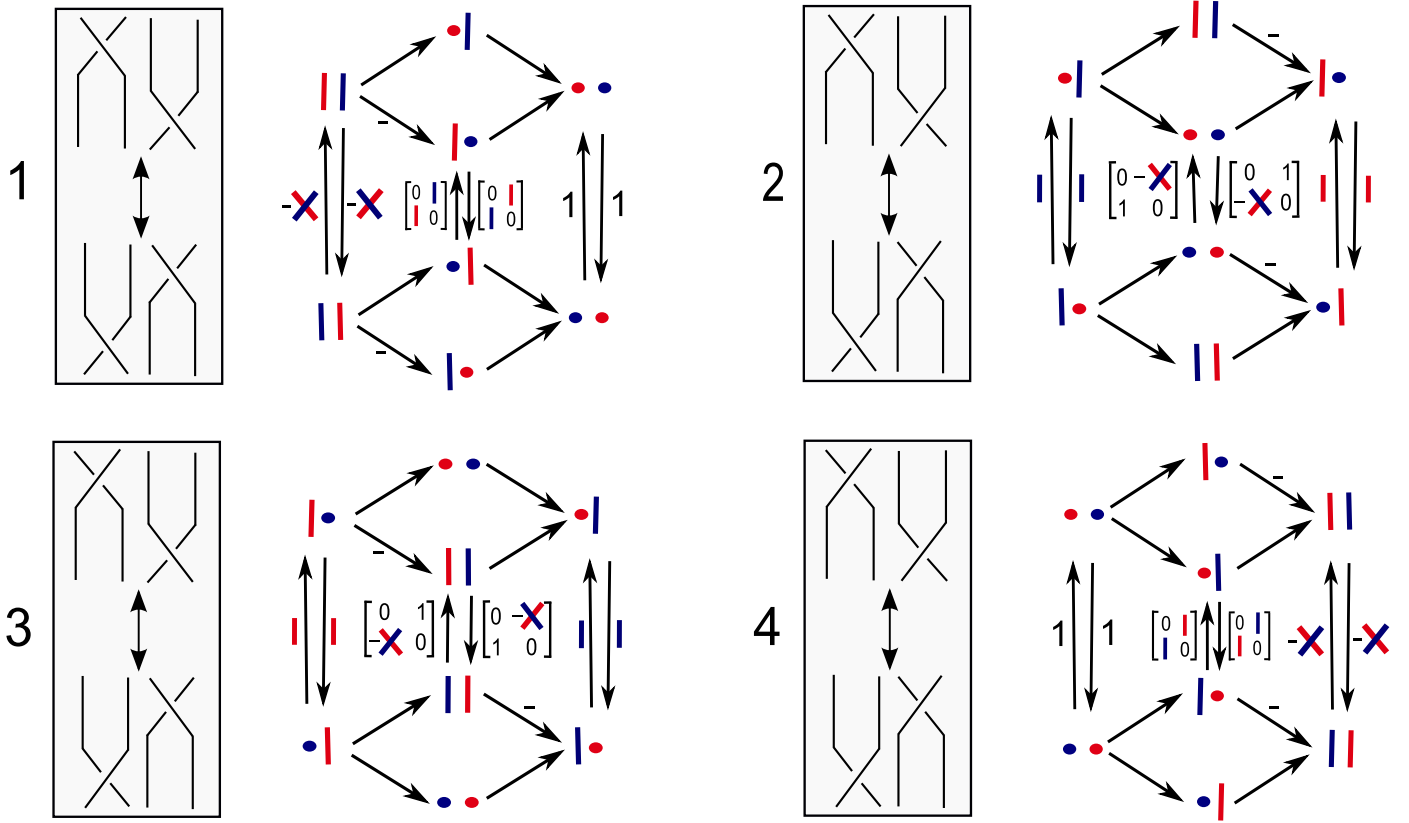


Figure 6: Slide generators

- **Reidemeister 3 generators** There are 12 generators in all: 6 possibilities for the height orders of the 3 strands (denoted by a number 1 through 6), and two directions for the movie (denoted "a" or "b"). Thankfully, the color-switching symmetries of the Soergel calculus allow us explicitly list only 6. The left-hand column lists the generators, and the chain complexes they correspond to; switching colors in the complexes yields the corresponding generator listed on the right. Each of these variants has a free parameter x , and the parameter used for each variant is actually independent from the other variants.

Remark. Using sequences of R2-type generators and various movie moves we could have abstained from ever defining certain R3-type variants or proving the movie moves that use them. We never use this fact, and list all here for completeness.

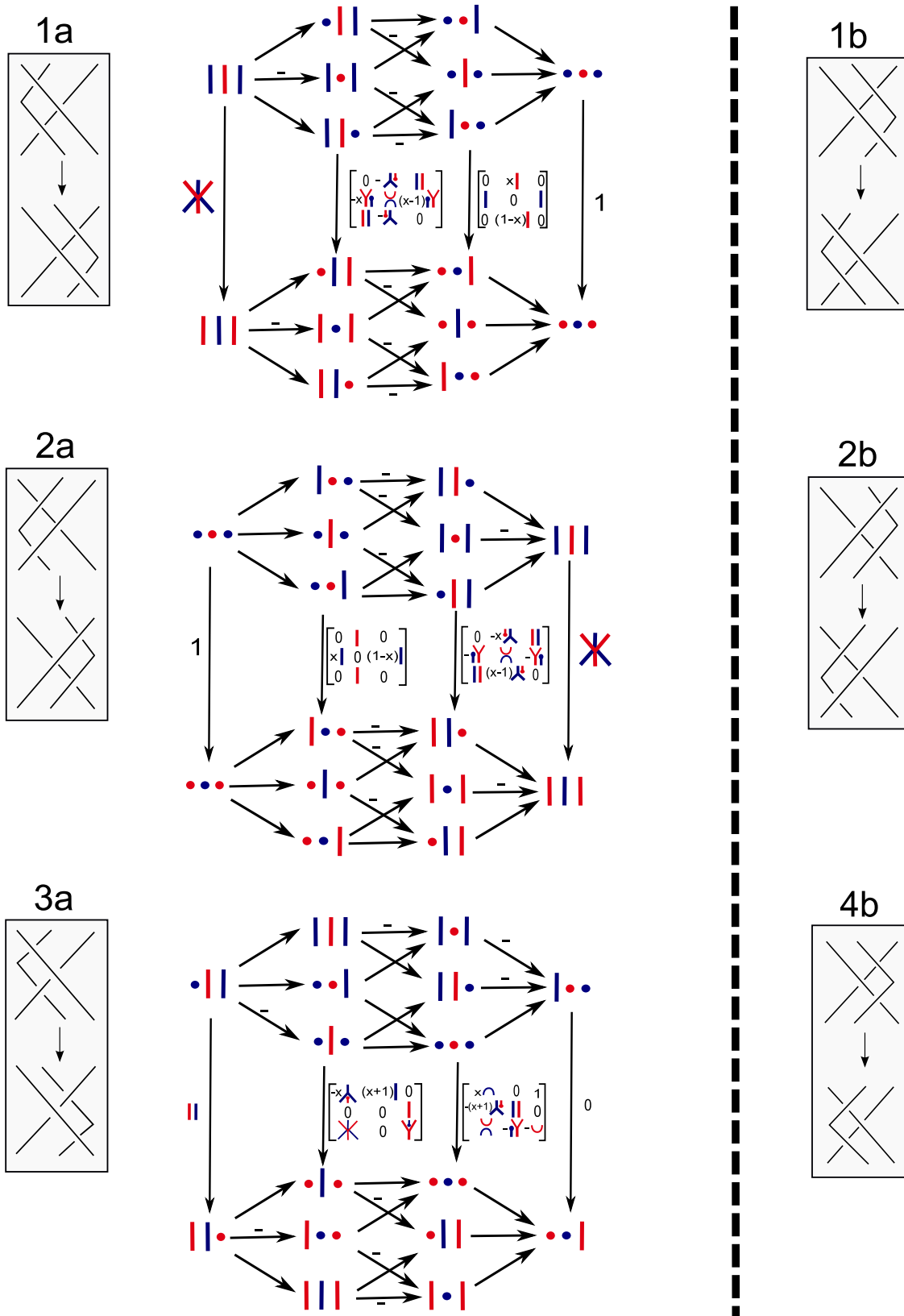


Figure 7: Reidemeister 3 type movie move generators

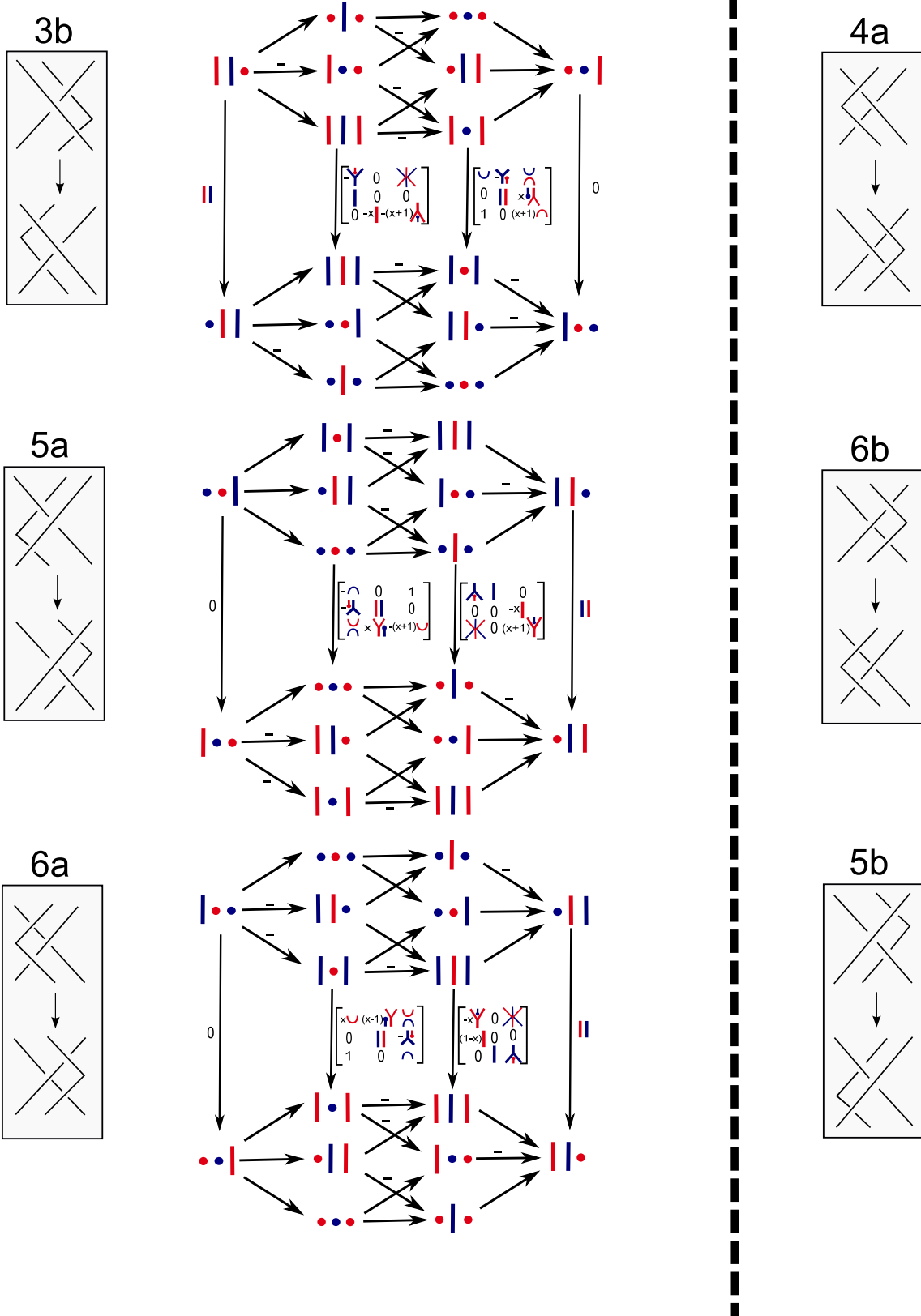


Figure 8: Reidemeister 3 type movie move generators

Claim 5. *Up to homotopy, each of the maps above is independent of x .*

Proof. *We prove the claim for generator 1a above; all the others follow from essentially the same computation. One can easily observe that there are very few summands of the source complex which admit degree -1 maps to summands of the target complex. In fact, the unique (up to scalar) non-zero map of homological degree -1 and graded degree -1 is a red trivalent vertex: a red fork which sends the single red line in the second row of the source complex to the double red line in the second row of the target complex. Given two chain maps, one with free variable x and one with say x' , the homotopy is given by the above fork map, with coefficient $(x - x')$. The homotopies for the other variants are exactly the same, save for the position, color, and direction of the fork (there is always a unique map of homological and graded degree -1).*

Remark. For all movie generators, there is a summand of both the source and the target which is B_\emptyset . We have clearly used the convention that for Type I movie generators, the induced map from the B_\emptyset summand in the source to the B_\emptyset summand in the target is the identity map. It is true that, with this convention, the chain maps above are the unique chain maps which would satisfy the movie move relations, where the only allowable freedom is given by the choice of various parameters x (exercise). There is no choice up to homotopy, so this is a unique solution.

Remark. Ignoring this convention, each of the above maps may be multiplied by an invertible scalar. Some relations must be imposed between these scalars, which the reader can determine easily by looking at the movie moves (each side must be multiplied by the same scalar). Movie move 11 forces all slide generators to have scalar 1. Movie move 13 forces all R3 generators to have scalar 1. Movie move 14 and 2 combined force the scalar for any R2 generator to be ± 1 , and then movie moves 2 and 5 force this sign to be the same for all 4 variants. Movie move 12 shows that the scalar for the birth of an overcrossing and the death of an undercrossing are related by the sign for the R2 generator. So the remaining freedom in the definition of the functor is precisely a choice of one sign and one invertible scalar.

4 Checking the Movie Moves

4.1 Simplifications

Given that the functor F has been defined explicitly, checking that the movie moves hold up to homotopy can be done explicitly. One can write down the chain maps for both complexes, and either check that they agree, or explicitly find the homotopy which gives the difference. This is not difficult, and many computations of this form were done as sanity checks. However, there are so many variants of each movie move that writing down every one would take far too long.

Thanks to Morrison, Walker, and Clark [2], a significant amount of work can be bypassed using a clever argument. The remainder of this section merely repeats results from that paper.

Notation 6. Let P, Q, T designate braid diagrams. $\text{Hom}(P, Q)$ will designate the hom space between $F(P), F(Q)$ in the homotopy category of complexes in \mathcal{SC}_2 . We write HOM for the graded vector space of all morphisms of complexes (not necessarily in degree 0). $\text{Hom}(B_{\underline{i}}, B_{\underline{j}})$ will still designate the morphisms in \mathcal{SC}_1 . Let $\mathbb{1}$ designate the crossingless braid diagram.

Lemma 7. (see [2]) Suppose that Movie Move 2 holds. Then there is an adjunction isomorphism $\text{Hom}(PO_i, Q) \rightarrow \text{Hom}(P, QU_i)$, or more generally $\text{Hom}(PT, Q) \rightarrow \text{Hom}(P, Q\overline{T})$. Similarly for other variations: $\text{Hom}(O_iP, Q) \rightarrow \text{Hom}(P, U_iQ)$, $\text{Hom}(P, QO_i) \rightarrow \text{Hom}(PU_i, Q)$, etc.

Proof. Given a map $f \in \text{Hom}(PO_i, Q)$, we get a map in $\text{Hom}(P, QU_i)$ as follows: take the R2 movie from P to PO_iU_i , then apply $f \otimes \text{id}_{U_i}$ to QU_i . The reverse adjunction map is similar, and the proof that these compose to the identity is exactly Movie Move 2. \diamond

Corollary 8. For any braid P , $\text{Hom}(P, P) \cong \text{Hom}(\mathbb{1}, P\overline{P})$.

Note that in the braid group, $P\overline{P} = \mathbb{1}$.

Lemma 9. Suppose that Movie Moves 3, 5, 6, and 7 hold. Then if P and Q are two braid diagrams which are equal in the braid group, then $\text{Hom}(P, T) \cong \text{Hom}(Q, T)$.

Proof. If two braid diagrams are equal in the braid group, one may be obtained from the other by a sequence of R2, R3, and distant crossing switching moves. Put together, these movie moves imply that all of the above yield isomorphisms of complexes. Thus P and Q have isomorphic complexes. \diamond

Remark. Technically, we don't even need these movie moves, only the resulting isomorphisms, which were already shown by Rouquier. However, since these movie moves are easy to prove and we desired the proofs in this paper to be self-contained, we show the movie moves directly.

Now the complex associated to $\mathbb{1}$ is just B_\emptyset in homological degree 0. So $\text{HOM}(\mathbb{1}, \mathbb{1}) = \text{HOM}(B_\emptyset, B_\emptyset)$, which we have already calculated is the free polynomial ring generated by double dots. In particular, the degree 0 morphisms are just multiples of the identity. Remember, this is a non-trivial fact in the graphical context! We will say more about this in Section 5.1.

Putting it all together, we have

Corollary 10. Suppose that Movie Moves 2, 3, 5, 6, 7 all hold. If P and Q are braid diagrams which are equal in the braid group, then $\text{Hom}(P, Q) \cong \mathbb{k}$, a one-dimensional vector space.

The practical use of finding one-dimensional Hom spaces is to apply the following method.

Definition 11. (See [2]) Consider two complexes A and B in an additive \mathbb{k} -linear category. We say that a summand of a term in A is homotopically isolated with respect to B if, for every possible homotopy h from A to B , the map $dh + hd: A \rightarrow B$ is zero when restricted to that summand.

Lemma 12. *Let ϕ and ψ be two chain maps from A to B , such that $\phi \equiv c\psi$ up to homotopy, for some scalar $c \in \mathbb{k}$. Let X be a homotopically isolated summand of A . Then the scalar c is determined on X , that is, $\phi = c\psi$ on X .*

The proof is trivial, see [2]. The final result of this argument is the following corollary.

Corollary 13. *Suppose that Movie Moves 2,3,5,6,7 all hold. If P and Q are braid diagrams which are equal in the braid group, and ϕ and ψ are two chain maps in $\text{Hom}(P, Q)$ which agree on a homotopically isolated summand of P , then ϕ and ψ are homotopic.*

Proof. Because the Hom space modulo homotopy is one-dimensional, we know there exists a constant c such that $\phi \equiv c\psi$. The agreement on the isolated summand implies that $c = 1$. \diamond

Most of the movie generators are isomorphisms of complexes; only birth and death are not. Hence, Movie Moves 1 through 10 all consist of morphisms P to Q , for P and Q equal in the braid group. Finding a homotopically isolated summand and checking the map on that summand alone will greatly reduce any work that needs to be done. Of course, one must show Movie Moves 2,3,5,6,7 independently before this method can be used.

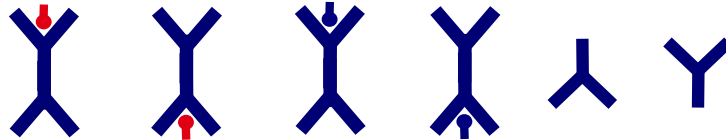
One final simplification, also found in Morrison, Walker and Clark, is that modulo Movie Move 8 all variants of Movie Move 10 are equivalent. Hence we can prove Movie Move 10 by investigating solely the overcrossing-only variant.

These simplifications apply to any functorial theory of braid cobordisms, so long as $\text{Hom}(\mathbb{1}, \mathbb{1})$ is one-dimensional. Now we look at what we can say specifically about homotopically isolated summands for Rouquier complexes in \mathcal{SC}_2 .

Any homotopy must be a map of degree -1 (if we ignore degree shifts on objects, as in our conventions). There are very few maps of negative degree in \mathcal{SC}_2 , a fact which immediately forces most homotopies to be zero. For instance, there are no negative degree maps from B_\emptyset to B_i , for any i . In an overcrossing-only braid, where B_\emptyset occurs in the maximal homological grading and various B_i show up in the penultimate homological grading, the B_\emptyset summand is homotopically isolated! Thus the overcrossing-only variant of Movie Move 10 will be easy. In fact, because of the convention we use that all isomorphism movie generators will restrict to multiplication by 1 from the B_\emptyset summand to the B_\emptyset summand, checking Movie Move 10 is immediate.

The only generators of negative degree are trivalent vertices. If each color appears no more than once in a complex, then there can be no trivalent vertices, so no homotopies are possible. This will apply to every variant of Movie Move 4, for instance.

Deducing possible homotopies is easy, as there are very few possibilities. For instance, the only nonzero maps which occur in homotopies outside of Movie Move 10 are:



We will not use these simplifications to their maximal effect, since some checks are easy enough to do without. For a discussion of other implications of checking the movie moves by hand, see Section 5.1.

4.2 Movie Moves

Note: (Logical sequence in the proofs of the movie moves.) We list the movie moves in numerical order, as opposed to logical order of interdependence. To use the technical lemma about homotopically isolated summands we first need to check movie moves 2,3,5,6,7. The reader will see that we prove these through direct computation, relying on none of the other moves.

- **MM1** There are eight variants of this movie (sixteen if you count the horizontal flip, which is just a color symmetry), of which we present two explicitly here. The key fact is that every slide generator behaves the same way: chain maps on summands have either a color crossing with a minus sign, the identity map with a plus sign, or zero; these maps occur precisely between the only summands where they make sense and, hence, have the same signs on both sides of the movie. Reversing direction uniformly changes the sign on the cups or caps in the R2 move. The only interesting part of the check uses a twist of relation (11). We describe in detail the movie associated to the first generator in figure 9, and give the composition associated to generator 3 in figure 10. Note that this check is trivial anyway since every summand is homotopically isolated.

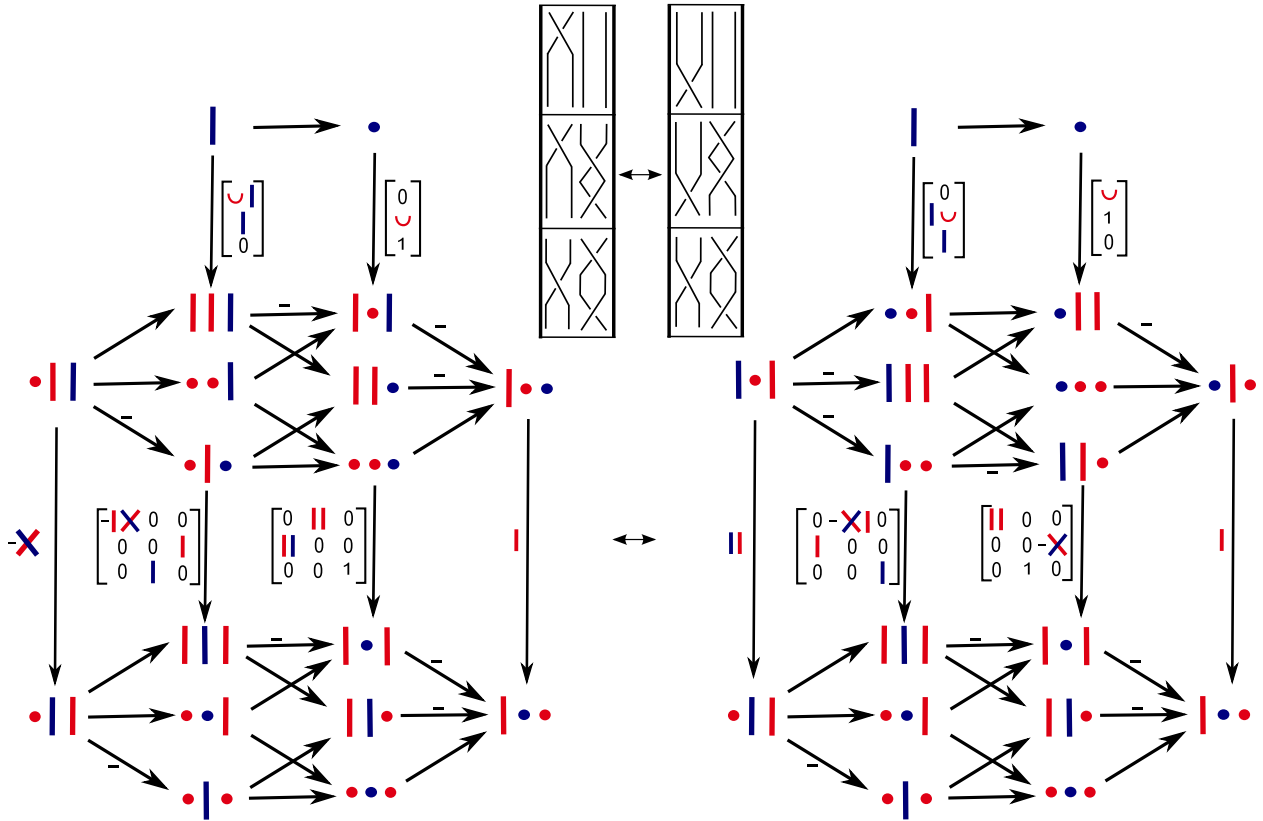


Figure 9: Movie Move 1 associated to slide generator 1

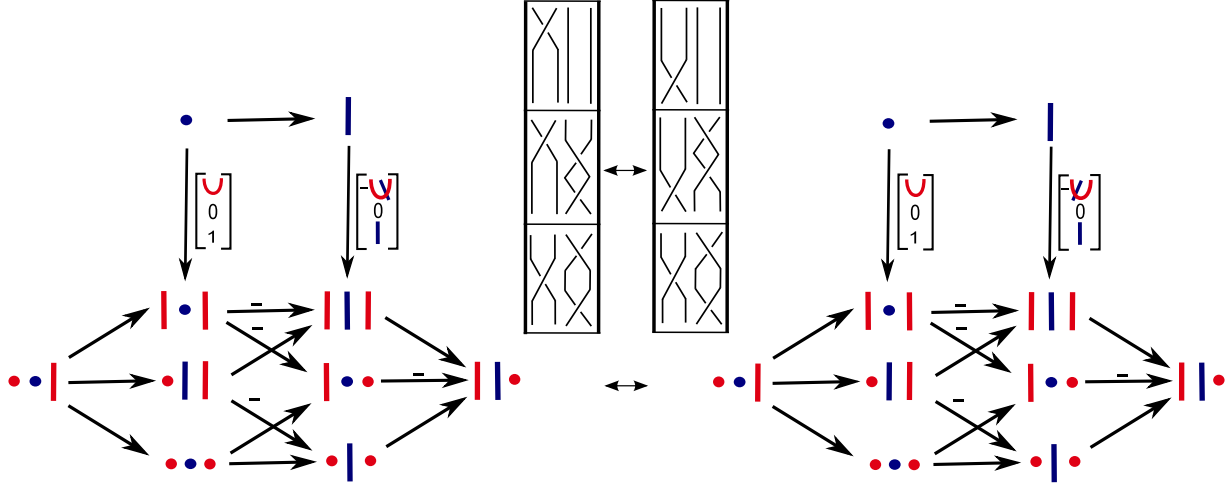


Figure 10: Movie Move 1 associated to slide generator 3

- **MM2** There are 4 variants to deal with here; we describe only one, and similar reasoning to that of MM1 will convince the reader that the other 3 are readily verified. The composition has the following form:

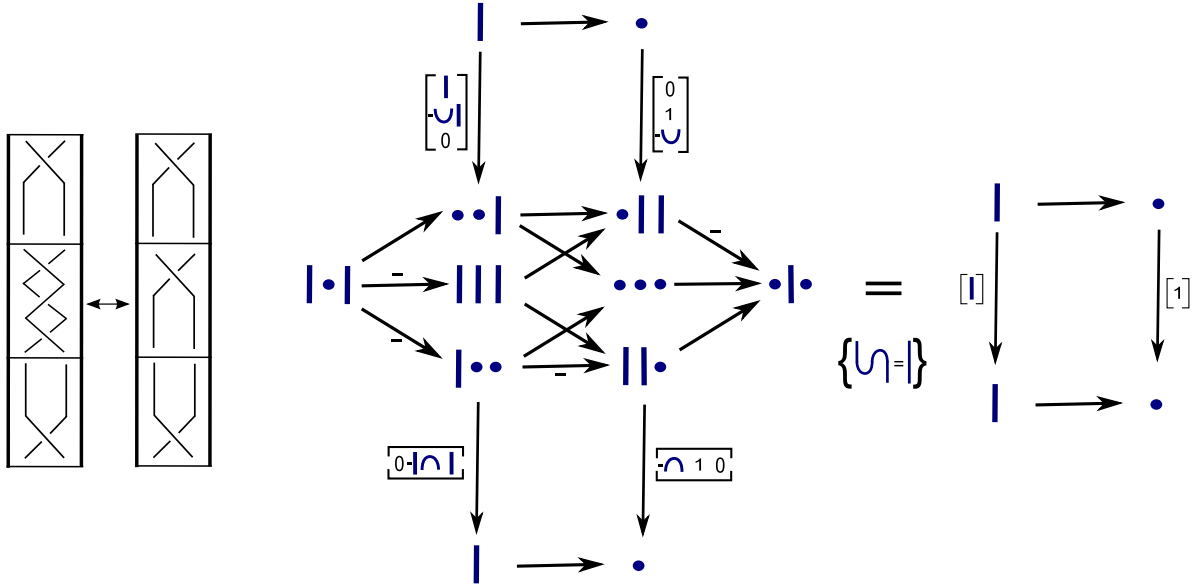


Figure 11: Movie Move 2

- **MM3** All 8 movie move 3 variants are essentially immediate after glancing at the slide generators, but we list one for posterity:

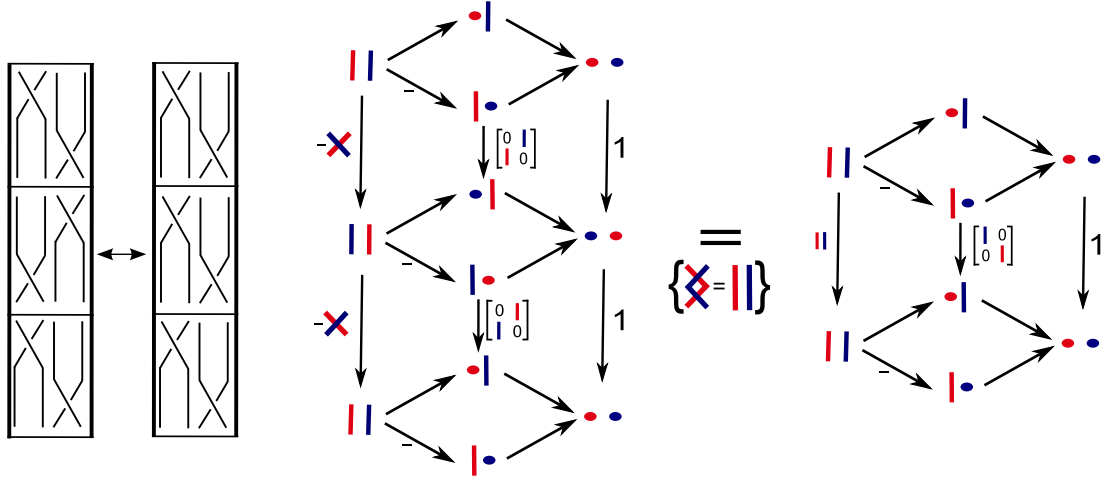


Figure 12: Movie Move 3

- **MM4** At this point the conscientious reader will find all 16 variants of movie move 4 quite easy, for the regularity of the slide chain maps allows one to write the compositions for the left and right-hand side at once. The maps only differ at the triple-color crossings, so we have to make use of relation (22).

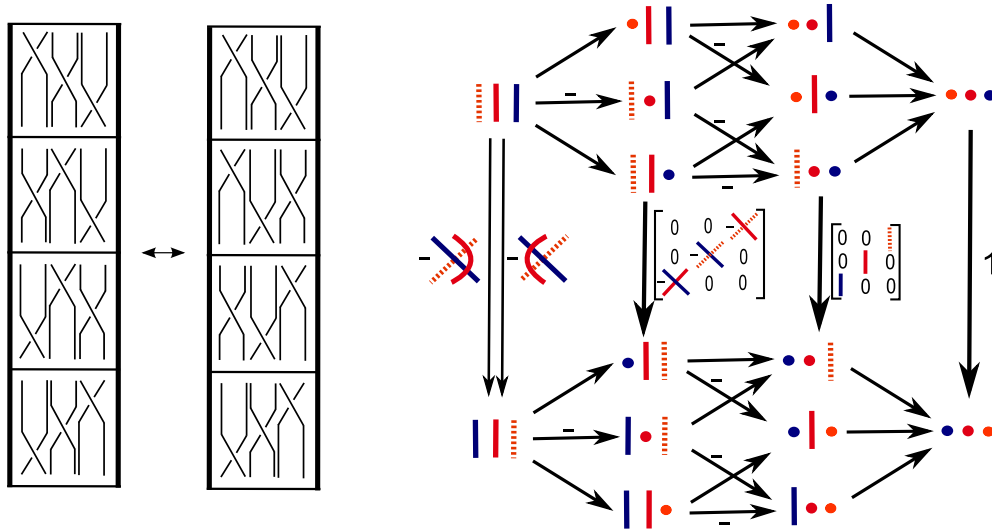


Figure 13: Movie Move 4

- **MM5** There are two variants of this movie, with the calculation for both almost identical. We consider the move associated to the first generator. The composition has the following form:

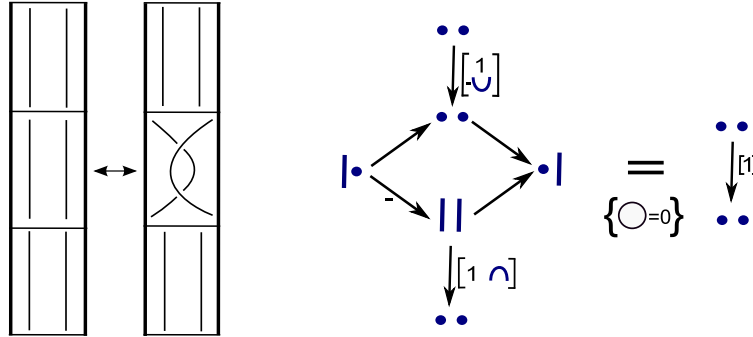


Figure 14: Movie Move 5

- **MM6** Again there are two variants and the calculation is almost as easy as the one for MM5; the only difference is that here we actually have to produce a homotopy. We check the variant associated to generator 1; left arrows are the identity, right the composition, and dashed the homotopy. Checking that the homotopy works requires playing with relation (16).

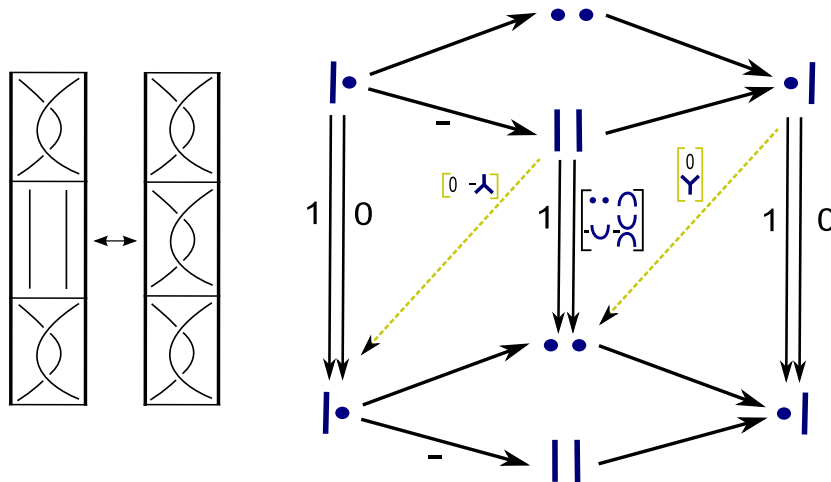


Figure 15: Movie Move 6

- **MM7** There are 12 variants of MM7, one for each R3 generator, and color symmetry will immediately reduce the number of different checks to 6; nevertheless, this is still a bit a drudge as each one requires a homotopy and a minor exercise in the relations. We display the movie associated to generator 1a and leave it to the very determined reader to repeat a very similar computation the remaining 5 times. The chain maps for the left-hand side of the movie are the following:

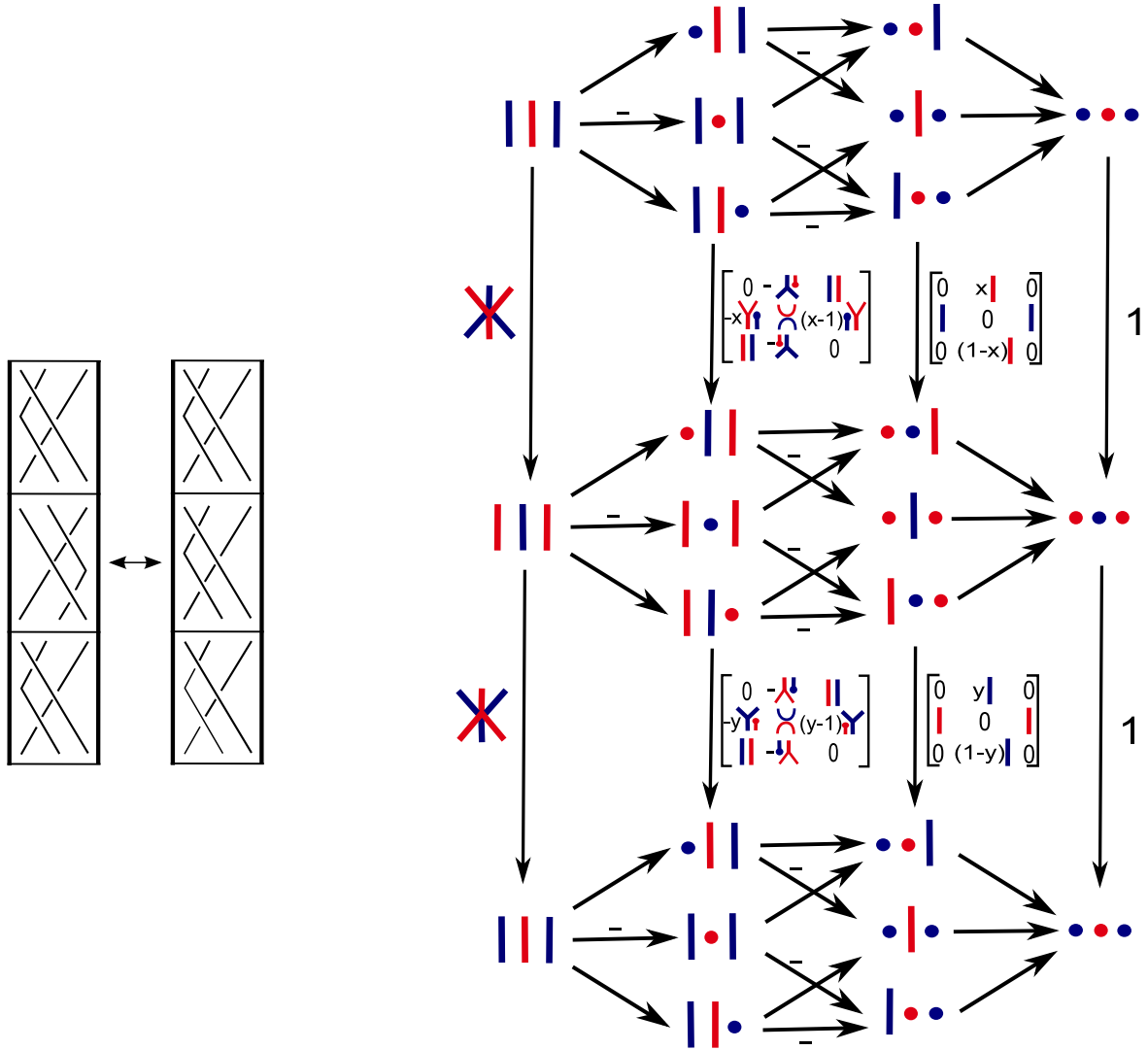


Figure 16: Movie Move 7

The composition and homotopy is:

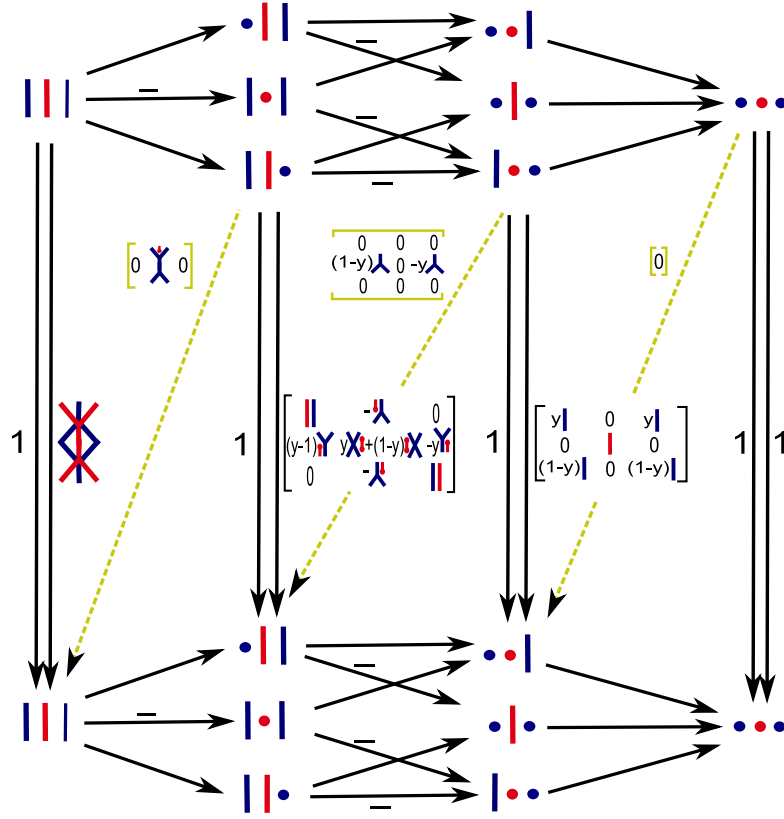


Figure 17: Homotopy for Movie Move 7

To check that the prescribed maps actually give a homotopy between and composition and the identity still requires some manipulation. The verification for the left-most map is simply relation (24). The verification for the right-most map is immediate, and for the third map is simple. This leaves us with the second map. Here $dH + Hd =$

$$\begin{bmatrix} 0 & \text{crossing} & 0 \\ 0 & -\text{crossing} & 0 \\ 0 & \text{crossing} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ (1-y)\text{crossing} & (1-y)\text{crossing} + y\text{crossing} & y\text{crossing} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \text{crossing} & 0 \\ (1-y)\text{crossing} & (1-y)\text{crossing} + y\text{crossing} - \text{crossing} & y\text{crossing} \\ 0 & \text{crossing} & 0 \end{bmatrix}$$

which save for the central entry is precisely the identity minus the composition. Equality of the central entry follows from this computation:

$$\begin{aligned} \text{crossing} - y\text{crossing} - (1-y)\text{crossing} &= \text{crossing} + \text{crossing} - \text{crossing} - \text{crossing} + y\text{crossing} - y\text{crossing} - y\text{crossing} + y\text{crossing} \\ &= \text{crossing} + \text{crossing} - \text{crossing} - \text{crossing} - y\text{crossing} + y\text{crossing} + y\text{crossing} - y\text{crossing} = (1-y)\text{crossing} + y\text{crossing} - \text{crossing} \end{aligned}$$

Note: This computation was done using relation (26) numerous times.

- **MM8** There are twelve variants of MM8: 3! possibilities for height order, and two directions the movie can run. All twelve are dealt with by the same argument, using

a homotopically isolated summand. There are no degree -1 maps from B_0 to any summand in the target, since there are at most two lines of a given color in the target, so we can assume there are no trivalent vertices. Hence the B_0 summand of the source is homotopically isolated, so we need only keep track of the homological degree 0 part, which significantly simplifies the calculation. We present one variant in diagram 18.

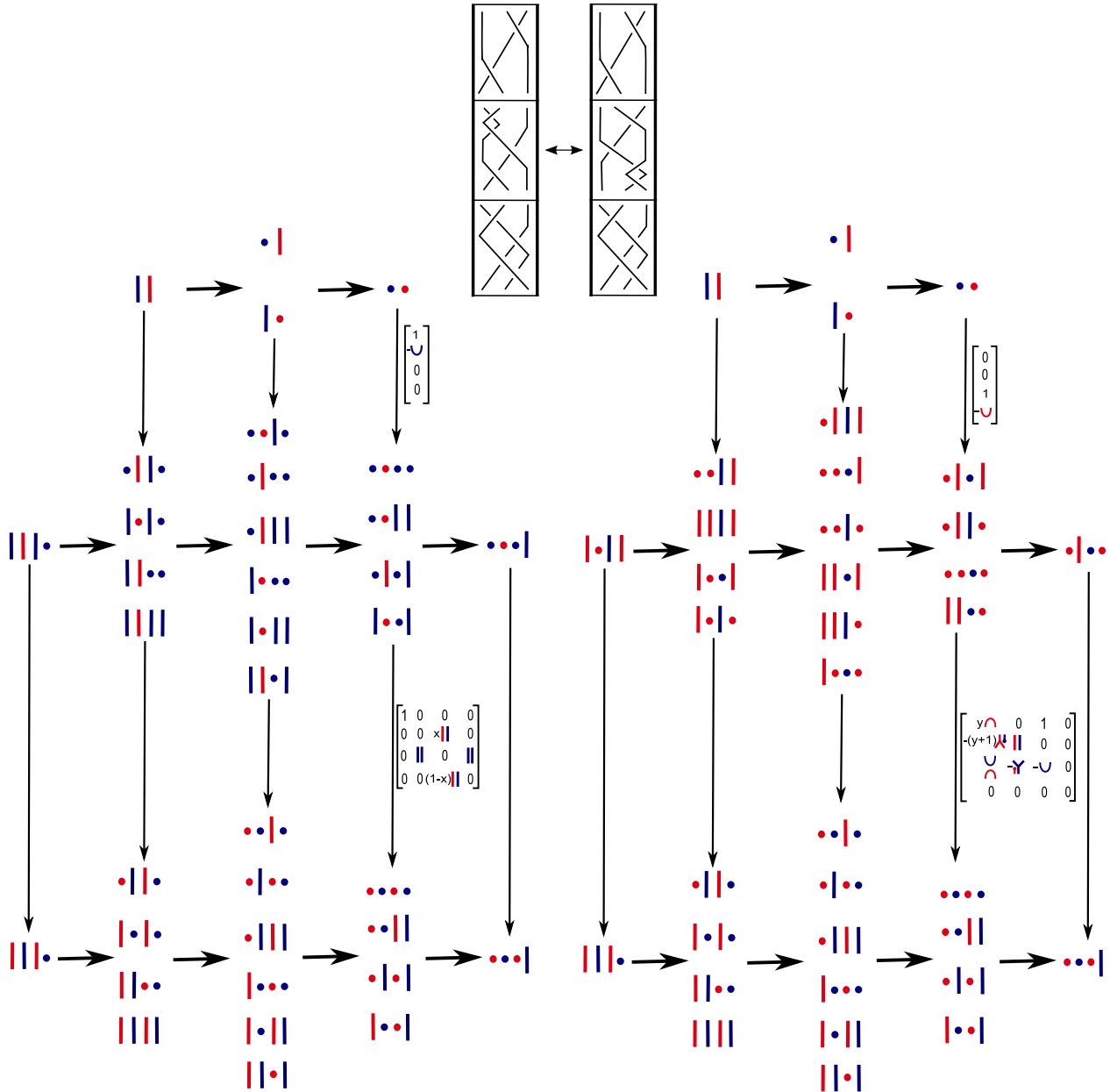


Figure 18: Movie Move 8

Composing the chain maps for the two sides of MM8 we see that they agree on B_0 .

- **MM9** There are a frightful 96 versions of MM9, coming from all the different R3 moves that can be done (12 in all), the type of crossing that appears in the slide, and horizontal and vertical flips. Once again, homotopically isolated summands come to the rescue. Again, in each variant there are no more than two crossings of a given color, so all maps from B_\emptyset to each summand in the target have non-negative degree. Thus the B_\emptyset summand of the source is homotopically isolated. Three colors are involved, the distant color and two adjacent colors. In the B_\emptyset summand, the distant-colored line does not appear, and no application of a distant slide or R3 move can make it appear. When the distant-colored line does not appear, the distant slide move acts by the identity. Thus both the right and left sides of the movie act the same way on the B_\emptyset summand, namely, they perform the R3 operation to it (sending it to the appropriate summands of the target).

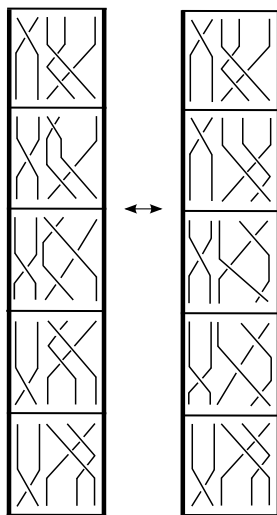


Figure 19: Movie Move 9

- **MM10** The sheer burden of writing down the complexes and calculating the chain maps for even one version of MM10 is best avoided at all costs. Despite at first seeming the more complicated of the movie moves, it is in the end the easiest to verify. We begin noting that, once one has shown MM8, all of the versions of MM10 are equivalent (see section 3.2.2 in [2]). So let us consider the variant with all left crossings. We see immediately that the B_\emptyset summand is homotopically isolated, that it is the unique summand in homological degree 0 in every intermediate complex, and that the chain maps all act by the identity in homological degree 0. Hence both sides agree on a homotopically isolated summand.

- **MM11** There are 32 variants of MM11: 2 choices of crossing, a vertical and a horizontal flip, and the direction of the movie. Half of these have chain maps that compose to zero on both sides, since the birth of a right crossing or the death of a left crossing is the zero chain map. The rest are straightforward. We give an example below in figure 20.

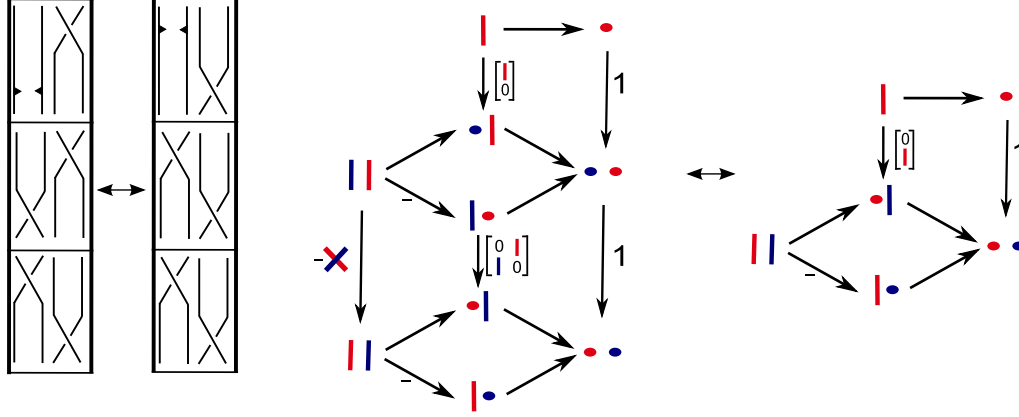


Figure 20: Movie Move 11

- **MM12** There are 8 variants: a choice of R2 move, a vertical flip, and the direction of the movie. Again, half of these are zero all around. Here are two variants; the other two are extremely similar.

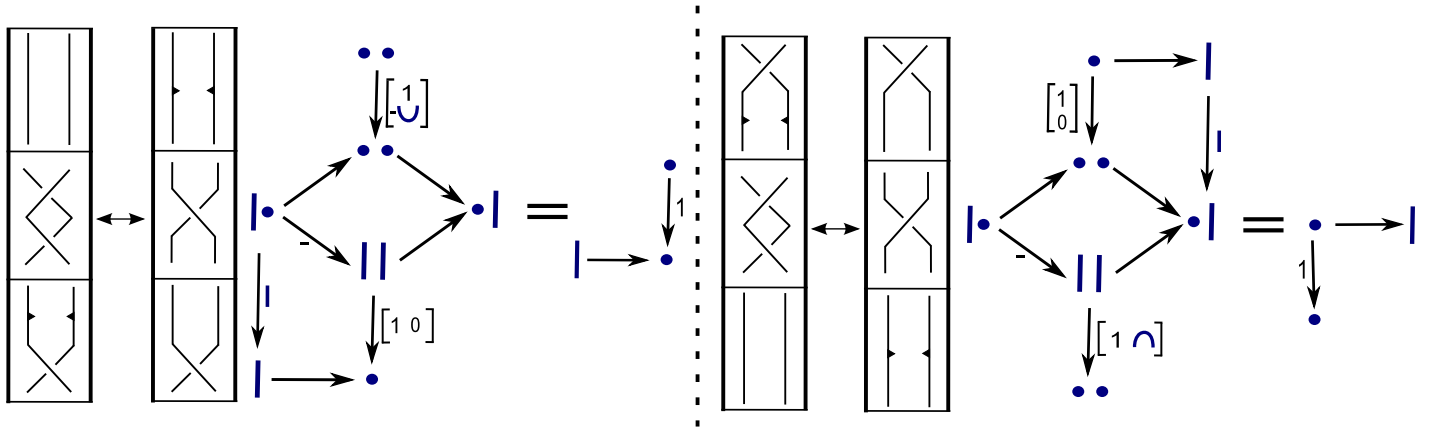


Figure 21: Movie Move 12

- **MM13** There are 24 variants: 12 R3 generators and two directions. Half are zero, and color symmetry for R3 generators reduces the number to check by half again. For the 6 remaining variants, the check requires little more than just writing down the composition, since the required homotopy in each instance is quite easy to guess. In figure 22 we describe the variant associated to the first R3 generator.

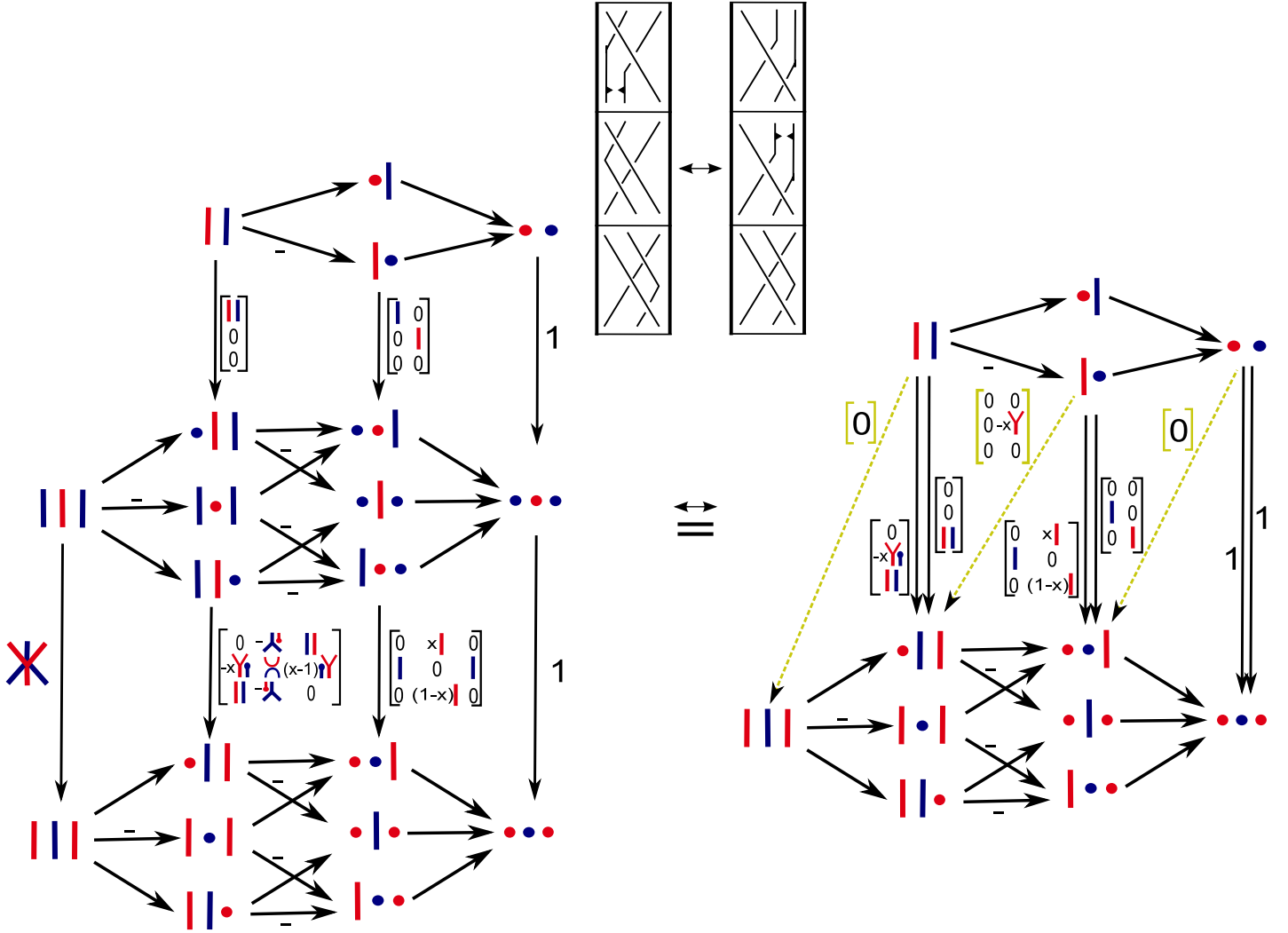


Figure 22: Movie Move 13

- **MM14** Since none of the R3 generators of type 1 or 2 is compatible with MM14, we are left with 16 variants: 8 R3 generators and 2 directions. As usual, there are only 4 to check. In addition to this, the initial frame of the movie corresponds to a complex supported in homological degree 0 only, so we only need write down what happens there. In figure 23 we describe the variant associated to the R3 generator 3a.

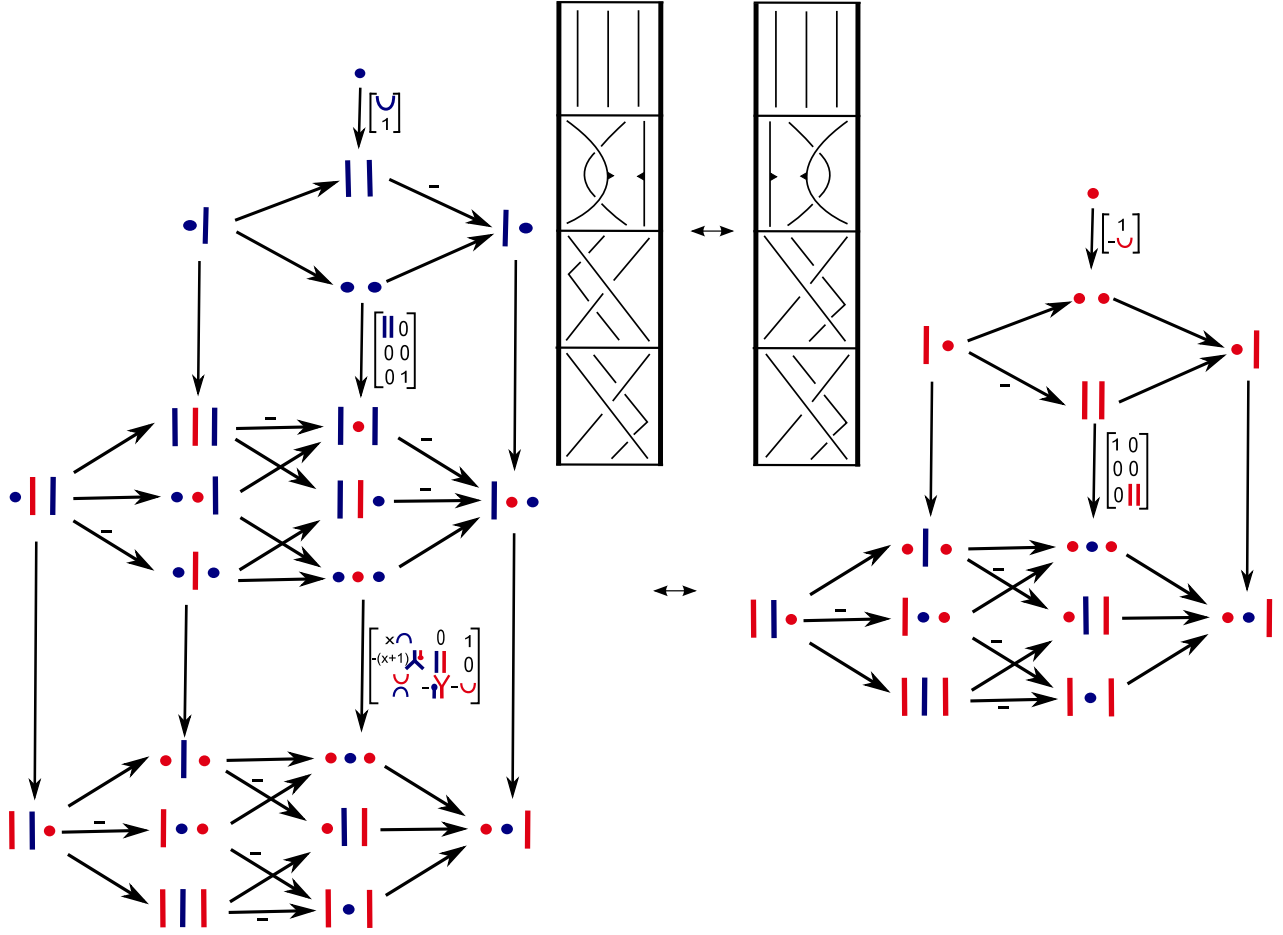


Figure 23: Movie Move 14

5 Additional Comments

5.1 The Benefits of Brute Force

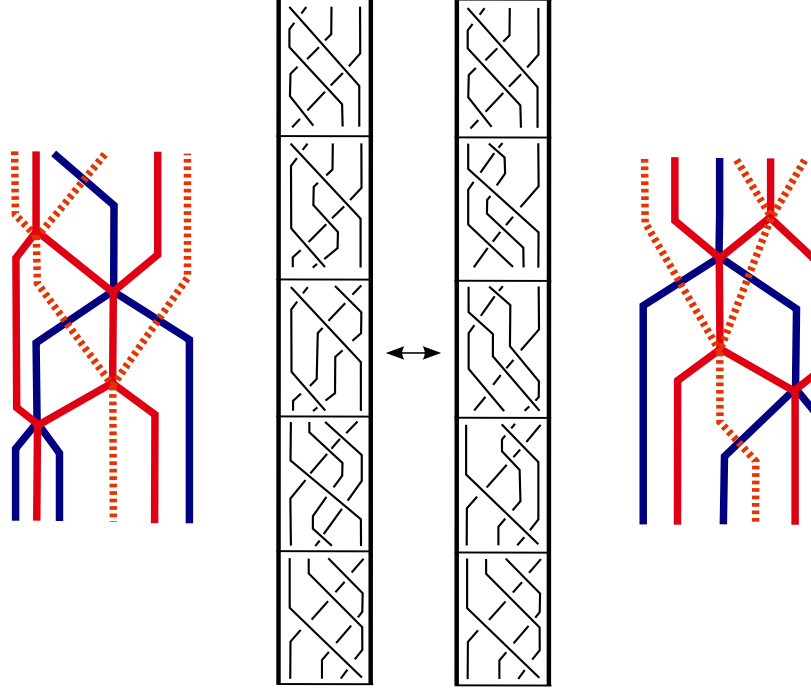
We have now shown that there is a functor from the braid cobordism category into the homotopy category of complexes in \mathcal{SC}_2 . Our method of proof used homotopically isolated summands, and hence relied on the fact that $\text{Hom}(B_\emptyset, B_\emptyset)$ was 1-dimensional. This is a trivial fact in the context of R -bimodules, amounting to the statement that $\text{HOM}(R, R) = R$. However, it is a non-trivial fact to prove for the graphical definition of \mathcal{SC}_1 , requiring the more complicated graphical proofs in [3]. Moreover, $\text{Hom}(B_\emptyset, B_\emptyset)$ need not be 1-dimensional in some arbitrary category \mathcal{C} of which \mathcal{SC}_1 is a (non-full) subcategory, and we may be interested in such categories \mathcal{C} . For instance, it would be interesting to define such a category \mathcal{C} for which one would have all birth and death maps nontrivial (although the authors have yet to find an *interesting* extension of this type).

Our method of proof, however, is irrelevant and the truth of Theorem 1 does not depend on it. One could avoid any machinery by checking each movie move explicitly (in fact, the only ones that remain to be checked are MM8, MM9, and MM10). Checking even a single variant of MM10 by brute force is extremely tedious, since each complex has 64 summands, but it could be done. In addition, we have actually proven slightly more: for any additive monoidal category \mathcal{C} having objects B_i and morphisms satisfying the \mathcal{SC}_1 relations, we can define a functor from the braid cobordism category into the homotopy category of complexes in \mathcal{C} . This is an obvious corollary, since that same data gives a functor from \mathcal{SC}_2 to \mathcal{C} . If one chose to change the birth and death maps, the proof for movie moves 1 through 10 would be unchanged, and one would only need to check 11 through 14.

One other benefit to (theoretically) checking everything by hand is in knowing precisely which coefficients are required, and thus understanding the dependence on the base ring \mathbb{k} . In all the movie moves we check in this paper, each differential, chain map, and homotopy has integral coefficients (or free variables which may be chosen to be integral). In fact, every nonzero coefficient that didn't involve a free variable was ± 1 , and free variables may be chosen such that every coefficient is 1, 0, or -1 . From our other calculations, the same should be true for MM8 through MM10 as well (Khovanov and Thomas [6] already showed that Rouquier complexes lift over \mathbb{Z} to a projective functor, which implies the existence of homotopy maps over \mathbb{Z}). The next section discusses the definition of this functor in a \mathbb{Z} -linear category.

As an additional bonus, checking the movie moves does provide some intuition as to why \mathcal{SC}_1 has the relations that it does. One might wonder why these particular relations should be correct: in [3] we know they are correct because they hold in the R -bimodule category and because they are sufficient to reduce all graphs to a simple form. There should be a more intuitive explanation.

As an illustrative example, consider the overcrossing-only variation of Movie Move 10 and the unique summand of lowest (leftmost) homological degree: it is a sequence of 6 lines. Then the left hand movie and the right hand movie correspond to the following maps on this summand:



Thus equality of these two movies on the highest term, modulo relation (17), is exactly relation (27).

Similarly, the highest terms in various other movie move variants utilize the other relations, as in the chart below.

MM	Relation
1	(11)
2	(8)
3	(17)
4	(22)
5	(15)
8	(12)
9	(21)
10	(27)

We can view these relations heuristically as planar holograms encoding the equality of cobordisms given by the movie moves.

More relations are used to imply that certain maps are chain maps, or that homotopies work out correctly. For example, relation (18) is needed for the slide generator to be a chain map. One can go even further, although we shall be purposely vague: so long as one disallows certain possibilities (like degree ≤ 0 maps from a red line to a blue line, or negative degree endomorphisms of indecomposable objects) then our graphical generators must exist a priori, and must satisfy a large number of the relations above.

Type II movie moves (11 through 14) do not contribute any relations or requirements not already forced by Type I movie moves (although they do fix the sign of various generators).

Almost every relation in the calculus is used in a brute force check of functoriality (including the brute force checks of MM8-10). However, there are two exceptions: (13) and

(25). Both these relations are in degree -2, and degree -2 does not appear in chain maps or homotopies, so they could not have appeared. Nonetheless, they are effectively implied by the remainder of the relations. It is not hard to use the rest of the one color relations to show that

$$\text{Crossing with dot on right} = \text{Crossing with dot on left}$$

Hence, (13) will hold, so long as R acts freely on morphisms. Under this mild assumption, all the relations are required. While no proof is presented here, it is safe to say that the category \mathcal{SC}_1 is universal amongst all categories for which Rouquier complexes could be defined functorially up to Type I movie moves (under suitable conditions on color symmetry and torsion-free double dot actions), and that these relations are effectively predetermined.

5.2 Working over \mathbb{Z}

Knot theorists should be interested in a \mathbb{Z} -linear version of the Soergel bimodule story, because it could theoretically yield a functorial link homology theory over \mathbb{Z} . We describe the \mathbb{Z} -linear version below. Because defining things over \mathbb{Z} is not really the focus of this paper, and because a thorough discussion would require poring over [3] for coefficients, we do not provide rigorous proofs of the statements in this section.

Ignoring the second equality in (26), which is equivalent to (16) after multiplication by 2, every relation given has coefficients in \mathbb{Z} . One could use these relations to define a \mathbb{Z} -linear version of \mathcal{SC}_1 and \mathcal{SC}_2 , and then use base extension to define the category over any other ring. The functor can easily be defined over \mathbb{Z} , as we have demonstrated, and all the brute force checks work without resorting to other coefficients. Theorem 1 still holds for the \mathbb{Z} -linear version of \mathcal{SC}_2 .

In fact, the same method of proof (using homotopically isolated summands) will work over \mathbb{Z} in most contexts. One begins by checking the isomorphisms (4) through (6). The only one which is in doubt is $B_i \otimes B_i \cong B_i\{-1\} \oplus B_i\{1\}$. So long as, for each i , there is an adjacent color in I , we may use (29) to check this isomorphism. Otherwise, we are forced to use (28), which does not have integral coefficients.

For now, assume that adjacent colors are present; we will discuss the other case below. One still has a map of algebras from \mathcal{H} to the additive Grothendieck group of \mathcal{SC}_1 . A close examination of the methods used in the last chapter of [3] will show that the graphical proofs which classify $\text{HOM}(\emptyset, \underline{i})$ still work over \mathbb{Z} in this context. Boundary dots with a polynomial will be a spanning set for morphisms. One can still define a functor into a bimodule category to show that this spanning set is in fact a basis. Therefore, the Hom space pairing on \mathcal{SC}_1 will induce a semi-linear pairing on \mathcal{H} , and it will be the same pairing as before. Hom spaces will be free \mathbb{Z} -modules of the appropriate graded rank, and this knowledge suffices to use all the homotopically isolated arguments.

Remark. This statement does *not* imply that \mathcal{SC} will categorify the Hecke algebra when defined over \mathbb{Z} . There may be missing idempotents, or extra non-isomorphic idempotents, so that the Grothendieck ring of the idempotent completion may be too big or small.

If adjacent colors are not present, the easiest thing to do to prove Theorem 1 is to include $\mathcal{SC}_1(I)$ into a larger $\mathcal{SC}_1(I')$ for which adjacent colors are present. Since this inclusion is faithful, all movie move checks which hold for I' will hold for I . Alternatively, one could use an extension of the category $\mathcal{SC}_1(I)$, extending the generating set by adding more polynomials, either as originally done in [3], or by formally adding $\frac{1}{2}$ times the double dot. Both of these should give an integral version of the category where the isomorphism (4) holds, and where the graphical proofs of [3] still work. Finally, if one does not mind ignoring 2-torsion, defining the category over $\mathbb{Z}[\frac{1}{2}]$ will also work.

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